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Simulation Study of the Bayesian and Non-Bayesian Estimation of a new Lifetime Distribution Parameters with Increasing Hazard Rate

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, a new distribution known as the Shifted Chris-Jerry (SHCJ) distribution is proposed. The proposition is motivated by the need to compare the efficiency of various classical estimation methods as well as the bayesian estimation using gamma prior at linear-exponential loss, squared error loss and generalized entropy loss functions. Some useful mathematical properties are derived. Single acceptance sampling plans (SASPs) are created for the distribution when the life test is truncated at a predetermined period. The median

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lifetime of the SHCJ distribution with pre-defined constants is taken as the truncation time. To guarantee that the specific life test is obtained at the defined risk to the user, the minimum sample size is required. For a particular consumer's risk, the SHCJ distribution's parameters, and the truncation time including numerical results are obtained. A simulation study is carried out for the bayesian and non-bayesian estimation of the parameters. Data on blood cancer patients is used to demonstrate the usefuleness of the proposed distribution.

Keywords: Acceptance sampling; Chris-Jerry distribution; estimation; shifted Chris-Jerry distribution.

1 Introduction

For both wider applicability and suitability reasons, shifted distributions are preferred to generic domain distributions since the former answer to specific data peculiarities. Succinctly, the minimum value of a data set is the lower limit of the variable domain hence modeling the data is tailor-made. Generally, in probability modeling of life phenomena, researchers often define the range of the values of the support variable in the interval $(0, \infty)$. However, this situation do not really hold as portrayed in the literature. Therefore, such models are not usually the exact representation of realities which consequently increase the degree of uncertainty in the inference made based on such models. The primary purpose of shifted distributions therefore is to take into consideration the uniqueness of every data set encountered in modeling. To this end, the shift parameter is used to adequately represent the actual initial boundary of the support variable in probability modeling. This dimension in modeling is recently receiving great attention in the statistical literature.

Chris-Jerry distribution credit to Onyekwere and Obulezi [1] is a one-parameter life time distribution that is gaining attention in statistical literature. This is because it fits variety of data sets better than many competing distributions such as Lindley distribution [2], Exponential distribution [3], Akash distribution [4], Aradhana distribution [5], Sujatha distribution [6], Ishita distribution [7], XGamma distribution [8], Rama distribution [9], Shanker distribution [12], Rani distribution [10] and Pranav distribution [11]. A lot of extensions have been proposed in the literature such as Modification of Shanker distribution using quadratic rank transmutation map [13],

Kumaraswamy Chris-Jerry distrbution by Obulezi et al. [14], Zubair-Exponential distribution [15], Exponentiated Power Lindley-Logarithmic distribution([16], [17]), Power size biased Chris-Jerry distribution [18], a new modified Lindley distribution by Chesneau [19], weibull distribution with estimable shift parameter by Onuoha et al. [20] and Marshall-Olkin Chris-Jerry distribution [21].

The basic motivation for this study is to

- proposed an extension of Chris-Jerry distribution with just an additional parameter called shift parameter that will account for the actual minimum value of any real life data of choice.
- a new Chris-Jerry distribution without the limitation associated with the generic lower boundary of data.

The rest of this paper is organized as follows; in section 2, we discuss the new distribution with its plots. In section 3, we derive some of its properties. In section 4, we develop the single acceptance sampling plans based on the truncated life tests with numerical results. In section 5, we discuss the classical estimation procedures. In section 6, we discuss the bayesian estimation method based on the linear-exponential loss, generalized entropy loss and the squared error loss functions. In section 7, we run a simulation study for the classical and bayesian estimation methods. In section 8, a real life data on blood cancer patients is used to illustrate the usefulness of the proposed distribution and the work is concluded in section 9.

2 The Suggested New Lifetime Distribution

Onyekwere and Obulezi [1] introduced the one-parameter lifetime distribution named Chris-Jerry (CJ) distribution, which has pdf and cdf given by

$$f_{CJ}(x,\theta) = \frac{\theta^2}{\theta+2}\left(1+\theta x^2\right)e^{-\theta x}; \quad x>0 \quad \theta>0 \tag{1}$$

and

$$F_{CJ}(x;\theta) = 1-\left[1+\frac{\theta x\,(\theta x+2)}{\theta+2}\right]e^{-\theta x} \tag{2}$$

Definition 2.1. Let $X \sim SHCJ(\theta,\gamma)$, the pdf and cdf are given by

$$f_{SHCJ}(x;\theta) \quad = \quad \frac{\theta^2}{\theta+2}\left(1+\theta\,(x-\gamma)^2\right)e^{-\theta(x-\gamma)}, x \geq \gamma, \theta>0, \gamma>0 \tag{3}$$

$$F_{SHCJ}(x;\theta) \quad = \quad 1-\left[1+\frac{\theta(x-\gamma)(\theta(x-\gamma)+2)}{\theta+2}\right]e^{-\theta(x-\gamma)}, x \geq \gamma, \theta>0, \gamma>0 \tag{4}$$

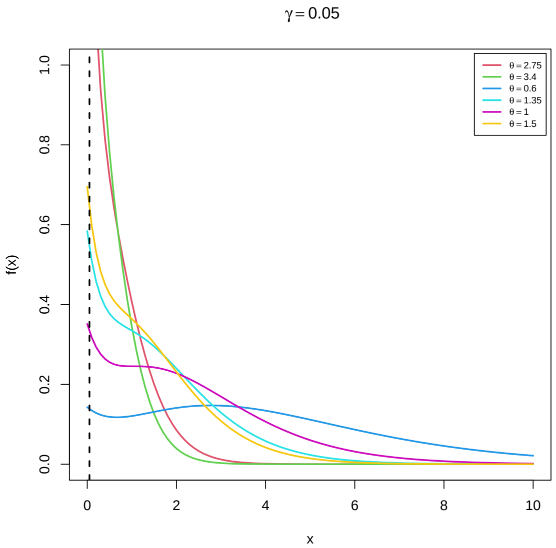

Fig. 1. pdf of the shifted Chris-Jerry dist

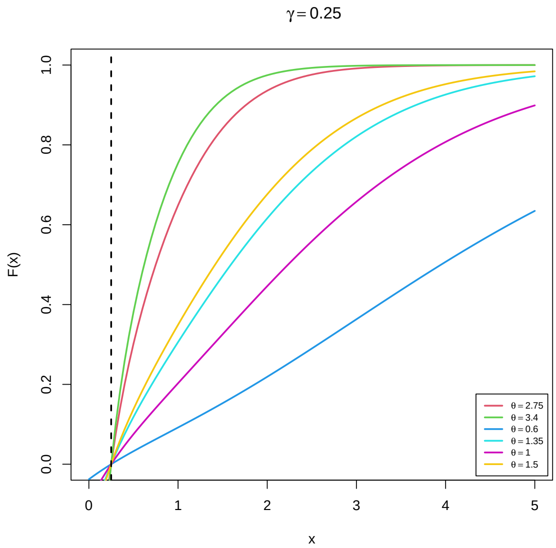

Fig. 2. cdf of the shifted Chris-Jerry dist

Definition 2.2. Let $X \sim SHCJ(\theta,\gamma)$, then the survival and hazard functions are given by

$$S(x;\theta,\gamma) = \left[1+\frac{\theta(x-\gamma)(\theta(x-\gamma)+2)}{\theta+2}\right]e^{-\theta(x-\gamma)}, x \geq \gamma, \theta>0, \gamma>0 \tag{5}$$

and

$$h(x;\theta,\gamma) = \frac{\theta^2\left(1+\theta\,(x-\gamma)^2\right)}{\theta+2+\theta(x-\gamma)(\theta(x-\gamma)+2)} \tag{6}$$

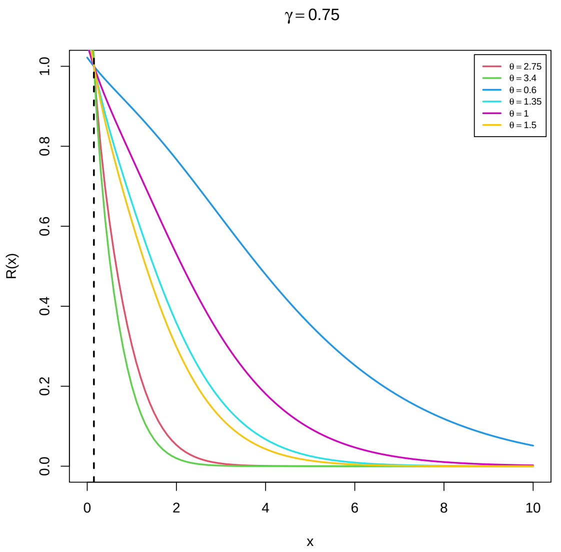

Fig. 3. Survival function of the SHCJ dist

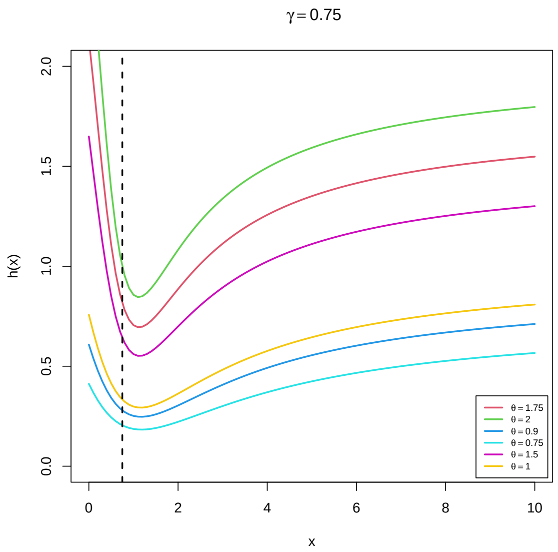

Fig. 4. Hazard function of the SHCJ dist

3 Distributional Properties

Definition 3.1 (Moment)**.** The r^{th} non-central moment of a Shifted Chris-Jerry random variable X is given as

$$\mu_k' = E(X^k) = \int_\gamma^\infty x^k f(x)dx = \int_\gamma^\infty x^k \frac{\theta^2}{\theta+2}(1+\theta(x-\gamma)^2)e^{\theta(x-\gamma)}dx = \frac{e^{\theta\gamma}k!}{\theta^k(\theta+2)}\left[\theta+\theta^2\gamma^2+(k+1)(k+2)+2\theta\gamma(k+1)\right] \quad (7)$$

Definition 3.2 (Mean)**.** The arithmetic mean is obtained from equation 3 above by substituting $r = 1$

$$\mu = \frac{e^{\theta\gamma}}{\theta(\theta+2)}\left[\theta+\theta^2\gamma^2+4\theta\gamma+6\right] \quad (8)$$

Definition 3.3 (Other useful non-central moments)**.** The 2^{nd}, 3^{rd} and 4^{th} non-central moments are obtained from equation 3 by substituting $r = 2, r = 3$ and $r = 4$ respectively

$$\mu_2' = \frac{2e^{\theta\gamma}}{\theta(\theta+2)}\left[\theta+\theta^2\gamma^2+6\theta\gamma+12\right] \quad (9)$$

$$\mu_3' = \frac{6e^{\theta\gamma}}{\theta(\theta+2)}\left[\theta+\theta^2\gamma^2+8\theta\gamma+20\right] \quad (10)$$

$$\mu_4' = \frac{24e^{\theta\gamma}}{\theta(\theta+2)}\left[\theta+\theta^2\gamma^2+10\theta\gamma+30\right] \quad (11)$$

Definition 3.4 (Useful central moments)**.** The 2^{nd}, 3^{rd} and 4^{th} central moments are respectively

$$\sigma^2 = \mu_2' - \mu^2 = \frac{e^{\theta\gamma} - e^{2\theta\gamma}}{\theta^2(theta+2)^2}\left[\theta^2 - 4\theta^3\gamma^3 + 4\theta^2\gamma + 16\theta - \theta^4\gamma^4 - 24\theta^2\gamma^2 - 24\theta\gamma + 12\right] \quad (12)$$

Definition 3.5 (Coefficient of variation).

$$\zeta = \frac{\sigma}{\mu} * 100$$
$$\frac{\sqrt{e^{-\theta\gamma}(\theta^2 - 4\theta^3\gamma^2 + 4\theta^2\gamma + 16\theta - \theta^4\gamma^4 - 24\theta^2\gamma^2 - 24\theta\gamma + 12)}}{e^{\theta\gamma}(\theta + \theta^2\gamma^2 + 4\theta\gamma + 6)} * 100 \quad (13)$$

Definition 3.6 (Coefficient of Dispersion).

$$\eta = \frac{\sigma^2}{\mu_1^1} = \frac{\theta^2 - 4\theta^3\gamma^2 + 4\theta^2\gamma + 16\theta - \theta^4\gamma^4 - 24\theta^2\gamma^2 - 24\theta\gamma + 12}{\theta(\theta+2)(\theta + \theta^2\gamma^2 + 4\theta\gamma + 6)} \quad (14)$$

Definition 3.7 (The Shape of Shifted Chris-Jerry Distribution: Mode). The mode of Shifted Chris-Jerry distribution is obtained by first taking the derivative of the pdf in equation 1

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{\theta^2}{\theta+2}\frac{d}{dx}(1+\theta(x-\gamma)^2)e^{-\theta(x-\gamma)} \\ &= \frac{\theta^2}{\theta+2}\left[-\theta e^{-\theta(x-\gamma)} - \theta^2(x-\gamma)^2 e^{-}\theta + 2\theta(x-\gamma)e^{-\theta(x-\gamma)}\right]\end{aligned} \quad (15)$$

It follows that for $\theta \leq 1$ then $\frac{d}{dx}f(x) = 0$

$$\begin{aligned}\frac{\theta^2}{\theta+2}\left[-\theta e^{-\theta(x-\gamma)} - \theta^2(x-\gamma)^2 e^{-\theta(x-\gamma)} + 2\theta(x-\gamma)e^{-\theta(x-\gamma)}\right] &= 0 \\ \theta(x-\gamma)^2 - 2(x-\gamma) + 1 &= 0\end{aligned} \quad (16)$$

the positive solution gives the mode, x_0 of the distribution

$$x_0 = \frac{\gamma(\theta+2) + 2\gamma^2(\theta+2)\sqrt{(\theta+2) - 1}}{\theta - 2} \quad (17)$$

Definition 3.8 (Quantile function). The q-quantile of Shifted Chris-Jerry distribution is obtained using $F(x_q) = P(X \leq x_q) = q$ for $0 < q < 1$.
Replace x with x_q in the cdf of Shifted Chris-Jerry distribution and equate to q

$$q = 1 - \left[1 + \frac{\theta(x_q-\gamma)(\theta(x_q-\gamma)+2)}{\theta+2}\right]e^{\theta(x-\gamma)} \quad (18)$$

$$(1-q)(\theta+2) = \left[\theta + 2 + \theta(x_q-\gamma)(\theta(x_q-\gamma)+2)\right]e^{\theta(x-\gamma)} \quad (19)$$

Solving the equation will give the quantile function x_q

Definition 3.9 (Stochastic Ordering of Shifted Chris-Jerry distribution). The stochastic ordering of a non-negative continuous random variable is a vital tool for comparing the behaviour of system components. A random variable X is said to be smaller than another random variable Y in the

Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ $\forall_x$ Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ $\forall_x$ Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ $\forall_x$ Likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{F_Y(x)}$ decreases in x

This implies that
$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y$

Theorem 1. Let $X \sim SHCJ(\theta_1)$ and $Y \sim SHCJ(\theta_2)$. IF $\theta_1 > \theta_2$ then $X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$

Proof.

$$\left.\begin{aligned}\frac{f_X(x)}{f_Y(x)} &= \frac{\frac{\theta_1^2}{\theta_1+2}(1+\theta_1(x-\gamma)^2)e^{-\theta_1(x-\gamma)}}{\frac{\theta_2^2}{\theta_2+2}(1+\theta_2(x-\gamma)^2)e^{-\theta_2(X-\gamma)}} \\ &= \frac{\theta_1^2(\theta_2+2)(1+\theta_1(x-\gamma)2)}{\theta_2^2(\theta_1+2)(1+\theta_2(x-\gamma)^2)}e^{-(\theta_2-\theta_1)(x-\gamma)}\end{aligned}\right\} \tag{20}$$

Taking natural log of the ratio will yeild

$$ln\frac{f_X(x)}{f_y(x)} = ln\frac{\theta_1^2(\theta_2+2)}{\theta} + \frac{(1+\theta_1(x-\gamma)^2)}{(1+\theta_2(x-\gamma)^2)} + (\theta_2-\theta_1)(x-\gamma) \tag{21}$$

Differentiating the natural log of the ratio wrt x will yield

$$\frac{d}{dx}ln\frac{f_X(x)}{f_y(x)} = \frac{2(x-\gamma)(\theta_1-\theta_2)}{(1+\theta_1(x-\gamma)^2)(1+\theta_2(x-\gamma)^2)} + (\theta_2-\theta_1) \tag{22}$$

If $\theta_2 > \theta_1, \frac{f_X(x)}{f_Y(x)} < 0$ and $\frac{f_X(x;\theta_1)}{f_Y(x;\theta_2)}$ is decreasing in x.

That is $X \leq_{lr} Y$ and hence $X \leq_{hr} Y, X \leq_{mrl} Y$ and $X \leq_{st} Y$ □

Definition 3.10 (Moment Generating Function of Shifted Chris-Jerry Distribution)**.** The MGF of Shifted Chris-Jerry

$$\begin{aligned}M_x(t) = E(e^{tx}) &= \int_0^\infty e^{tx}f(x)dx \\ &= \frac{\theta^2}{\theta+2}\int_0^\infty e^{tx}(1+\theta(x-\gamma)^2)e^{-\theta(x-\gamma)} \\ &= \frac{\theta^2 e^{\theta\gamma}}{\theta+2}\left[\frac{\Gamma}{(\theta-t)} + \frac{\theta\Gamma_3}{(\theta-t)^3} - \frac{2\gamma\theta\Gamma_2}{(\theta-t)^2} + \frac{\gamma\theta\Gamma}{(\theta-t)}\right] \\ &= \frac{\theta^2 e^{\theta\gamma}\left[(\theta-t)^{-1} + 2\theta(\theta-t)^3 - 2\gamma\theta(\theta-t)^{-2} + \gamma\theta(\theta-t)^{-1}\right]}{\theta+2}\end{aligned} \tag{23}$$

Definition 3.11 (Characteristics Function of Shifted Chris-Jerry Distribution)**.** The characteristics function of a $X \sim SHCJ(\theta)$ is given by

$$\begin{aligned}\phi_x(it) = E(e^{itx}) &= \int_0^\infty e^{itx}f(x)dx \\ &= \frac{\theta^2 e^{\theta\gamma}}{\theta+2}\left[\frac{1}{(\theta-it)} + \frac{2\theta}{(\theta-it)^3} - \frac{2\gamma\theta}{(\theta-it)^2} + \frac{\gamma\theta}{(\theta-it)}\right] \\ &= \frac{\theta^2 e^{\theta\gamma}\left[(\theta-it)^{-1} + 2\theta(\theta-it)^{-3} - 2\gamma\theta(\theta-it)^{-2} + \gamma\theta(\theta-it)^{-1}\right]}{\theta+2}\end{aligned} \tag{24}$$

Definition 3.12 (Distribution of the order Statistics)**.** Suppose $X_1, X_2, ..., X_n$ is a random sample of $X_{(r)}$; (r = 1,2,...,n) are the r^{th} order statistics obtained by arranging X_r in ascending order of magnitude $X_1 \leq X_2 \leq ... \leq X_r$ and $X_1 = min(X_1, X_2, ...X_r)$, $X_r = max(X_1, X_2, ...X_r)$ then the pdf of the r^{th} order statistic is given by

$$f_{r:n}(x,\theta) = \frac{n!}{(r-1)!(n-r)!}f(x,\theta)\left[F(x,\theta)\right]^{r-1}\left[1-F(x,\theta)\right]^{n-r} \tag{25}$$

where f(.) and F(.) are the pdf and cdf of SHCJ distribution respectively. Hence we have

$$f_{r:n}(x,\theta) = \frac{n!}{(r-1)!(n-r)!}\frac{\theta^2}{\theta+2}\left(1+\theta(x-\gamma)^2\right)e^{-\theta(x-\gamma)} \\ \times\left\{1-\left[1+\frac{\theta(x-\gamma)\left(\theta(x-\gamma)+1\right)}{\theta+2}\right]e^{-\theta(x-\gamma)}\right\}^{r-1}\left\{\left[1+\frac{\theta(x-\gamma)\left(\theta(x-\gamma)+1\right.}{\theta+2}\right]e^{-\theta(x-\gamma)}\right\}^{n-r} \tag{26}$$

The pdf of the largest order statistics is obtained by setting r=n

$$f_{n:n}(x,\theta) = \frac{n\theta^2}{\theta+2}\left(1+\theta(x-\gamma)^2\right)e^{-\theta(x-\gamma}\left\{1-\left[1+\frac{\theta(x-\gamma)\left(\theta(x-\gamma)+1\right)}{\theta+2}\right]e^{-\theta(x-\gamma)}\right\}^{n-1} \tag{27}$$

The pdf of the smallest order statistics is obtained by setting r=1

$$f_{1:n}(x,\theta) = \frac{n\theta^2}{\theta+2}\left(1+\theta(x-\gamma)^2\right)\left[1+\frac{\theta(x-\gamma)\left(\theta(x-\gamma)+1\right)}{\theta+2}\right]^{n-1}e^{-\theta n(x-\gamma)} \tag{28}$$

Definition 3.13 (Information measure and asympotic behaviour of Shifted Chris-Jerry distribution)**.** Entropy is the quantity of uncertainty or randomness in a system. It is an information measure for non-negative $\omega \neq 1$ .The Reny Entropy for Shifted Chris-Jerry distributed random variable X is

$$\begin{aligned}
R_\omega(x) &= \lim_{n\to\infty}\left(l_w(f_n) - logn\right)\\
&=\frac{1}{1-\omega}log\int_0^\infty f(x)^\omega dx\\
R_\omega(x) &=\frac{1}{1-\omega}log\int_0^\infty\left\{\frac{\theta^2}{\theta+2}\left(1+\theta(x-\gamma)^2\right)e^{-\theta(x-\gamma)}\right\}^\omega dx\\
&=\frac{1}{1-\omega}log\left[\frac{\theta^{2\omega}e^{\theta\omega\gamma}}{(\theta+2)^\omega}\sum_{k=0}^{\infty}\binom{\omega}{k}\theta^k\int_0^\infty(x-\gamma)^{2k}e^{-\theta\omega x}dx\right]\\
R_\omega(x) &=\frac{1}{1-\omega}log\left[\frac{\theta^{2\omega}e^{\theta\omega\gamma}}{(\theta+2)^\omega}\sum_{k=0}^{\infty}\sum_{j=0}^{2k}\binom{\omega}{k}\binom{2k}{j}\theta^k\gamma^{2k-j}\int_0^\infty x^je^{-\theta\omega x}dx\right]\\
&=\frac{1}{1-\omega}log\left[\frac{\theta^{2\omega}e^{\theta\omega\gamma}}{(\theta+2)^\omega}\sum_{k=0}^{\infty}\sum_{j=0}^{2k}\binom{\omega}{k}\binom{2k}{j}\theta^k\gamma^{2k-j}\frac{\Gamma_{j+1}}{(\theta\omega)^{j+1}}\right]
\end{aligned} \tag{29}$$

The asymptotic behaviour of the Shifted Chris-Jerry distributed random variable is investigated by taking the limit of the pdf as $x \to 0$ and as $x \to \infty$

$$\begin{aligned}
\lim_{x\to0}\frac{\theta^2}{\theta+2}\left(1+\theta(x-\gamma)^2\right)e^{-\theta(x-\gamma)} &=\frac{\theta^2}{\theta+2}\left(1+\theta\gamma^2\right)e^{-\theta\gamma}\\
\lim_{x\to\infty}\frac{\theta^2}{\theta+2}\left(1+\theta(x-\gamma)^2\right)e^{-\theta(x-\gamma)} &=\frac{\theta^2}{\theta+2}\lim_{x\to\infty}\left(1+\theta(x-\gamma)^2\right)e^{-\theta(x-\gamma)}
\end{aligned} \tag{30}$$

Definition 3.14 (Survival function and Failure function)**.** Given a continuous distribution with pdf and cdf in equations 1 and ‘2, the survival function is given by

$$S_{SCJ}(x;\theta) = 1 - F_{SCJ}(x;\theta) = \left\{1+\frac{\theta(X-\gamma(\theta(x-\gamma)+2)}{\theta+2}\right\}e^{-\theta(x-\gamma)}; x,\theta,\gamma \geq 0 \tag{31}$$

Notice that for Shifted Chris-Jerry distribution the survival function $S_{SCJ}(x;\theta) = 1$ as $x \to 0$ and $S_{SCJ}(x;\theta) = 0$ as $x \to \infty$. Also, the failure rate $h_{SHCJ}(x;\theta)$, an important tool in reliability measure and engineering is given by

$$h_{SHCJ}(x;\theta) = \frac{f_{SHCJ}(x;\theta)}{S_{SHCJ}(x;\theta)} = \frac{\theta^2(1+\theta(x-\gamma)^2)}{\theta+2\left(\theta(x-\gamma)(\theta(x-\gamma)+2\right)} \tag{32}$$

For the Shifted Chris-Jerry distribution, the failure rate exhibits the following behaviour;

$$h_{SHCJ}(0) = f_{SHCJ}(0) = \frac{\theta(1+\theta\gamma)}{\theta + 2(\theta\gamma(\theta\gamma+2))}$$

which is similar to the Lindley distribution. The function $h_{SHCJ}(x;\theta)$ is an increasing function in x and $\theta, h_{SHCJ}(\infty) = 0$

4 Single Acceptance Sampling Plans

Assume that a product's lifetime is based on the SHCJ distribution, which has the parameters (θ, γ) stated in equation 6, and that the producer's claimed industry standard for the lifetime of units is represented by M_0. The main goal is to determine if the proposed lot should be accepted or rejected based on the fact that the actual median life cycle of the units, m, is longer than the recommended lifetime, M_0. It is important to remember that it is standard procedure in life testing to end the test by the time indicated by T_0 and count the number of failures.

Singh and Yogesh [22] provided us with some guidelines on how to accept the proposed lot based on the evidence that $M \geq M_0$, given probability of at least α^* (consumer's risk), using a single acceptance sampling plan. The experiment is run for a $T_0 = M_0$ units of time, multiple of claimed median lifetime with any positive constant a. These are the actions:

1. Take n units at random from the proposed lot as a sample.
2. Run the following test for T_0 units of time:
   Accept the entire lot if c or fewer units (the acceptance number) fail throughout the experiment; else, reject the entire lot.

Be aware that the proposed sampling plan is given by and that the chance of accepting a lot considers suitably large lots to help with the application of the binomial distribution.

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 1, 2, \ldots, n, \tag{33}$$

where p is defined as $p = F_{SHCJ}(T_0; \theta, \gamma)$, according to equation 6. The sampling plan's operating characteristic function, or the acceptance probability of the lot as a function of the failure probability, is represented by the function $L(p)$. Using $T_0 = aM_0$, further, p_0 can be written as follows:

$$p_0 = F_{SHCJ}(T_0 = aM_0; \theta, \gamma) = 1 - \left[1 + \frac{\theta(T_0-\gamma)(\theta(T_0-\gamma)+2)}{\theta+2}\right] e^{-\theta(T_0-\gamma)} \tag{34}$$

Now, the problem is to determine for given values of $\alpha^* \, (0 < \alpha^* < 1), kM_0$ and c, the smallest positive integer n such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \leq 1 - \alpha^*, \tag{35}$$

where p_0 is given by equation 33.

The operating characteristic probability and the minimal values of n satisfying the inequality 34 are determined and shown in Table 1, Table 2, 3 and Table 4 for the following assumed parameters:

1. The consumer's risk α^* is given as: $0.30, 0.60,$ and 0.95.
2. The acceptance number c is given as: $0, 2, 4, 8,$ and 10.

3. The constant a is assumed to be: $0.10, 0.25, 0.50,$ and 0.75. If $a = 1$, thus T_0 is the median life time $M_0 = 0.5 \quad \forall \quad \theta, \gamma$).
4. The parameters (θ, γ) of the SHCJ distribution are assumed as:

$$\theta = (0.10, 0.20, 0.30, 0.40); \quad \gamma = 0.50$$

It is important to note that the choice of the sampling plans parameters are based on literature studies by Obulezi, Igbokwe and Anabike [23], Singh and Yogesh [22] and Gillariose [24].

From the results obtained in Table 1 to Table 4, we notice that:

- α^* and c increase, the sample size n increases leading to a decrease in the $L(p_0)$.
- As a increases, the required sample size n decreases and $L(p_0)$ is increases.
- Again, as θ increases and γ is fixed, the required sample size n increases and $L(p_0)$ decreases.

Finally, for all results we have obtained, we checked that $L(p_0) \leq 1 - \alpha^*$. Also, when $a = 1$, we have $p_0 = 0.5$ as $T_0 = M_0$ and hence all results $(n, L(p_0))$ for any vector of parameter (θ, γ) are the same.

Table 1. SASP for SHCJ distribution with parameter: $\gamma = 0.20$ for different values of θ

α^*	c	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\theta = 0.15$											
0.25	0	19	0.75135	8	0.75572	3	0.77613	1	1	1	1
	2	110	0.75319	45	0.75192	15	0.77227	5	0.85414	4	0.87500
	4	215	0.75030	87	0.75154	29	0.76495	10	0.79373	8	0.77344
	8	435	0.75106	175	0.75469	59	0.75105	20	0.76192	15	0.78802
	10	548	0.75131	221	0.75213	74	0.75216	25	0.75769	19	0.75966
0.75	0	88	0.25113	35	0.25656	11	0.28163	4	0.25065	3	0.25000
	2	248	0.25204	99	0.25609	32	0.26911	10	0.29349	7	0.34375
	4	398	0.25011	159	0.25406	52	0.25871	16	0.29523	12	0.27441
	8	685	0.25023	275	0.25003	90	0.25395	28	0.28244	21	0.25172
	10	825	0.25102	331	0.25147	108	0.25932	34	0.27441	25	0.27063
0.95	0	189	0.05049	75	0.05177	24	0.05423	7	0.06283	5	0.06250
	2	398	0.05023	159	0.05052	51	0.05335	15	0.06348	11	0.05469
	4	579	0.05016	231	0.05098	75	0.05108	22	0.06582	16	0.05923
	8	913	0.05034	365	0.05080	118	0.05323	36	0.05685	26	0.05388
	10	1073	0.05037	429	0.05091	139	0.05289	43	0.05113	30	0.06802
$\theta = 0.20$											
0.25	0	16	0.75169	7	0.75795	3	0.76336	1	1	1	1
	2	92	0.75393	39	0.75500	14	0.77951	5	0.85148	4	0.87500
	4	180	0.75014	75	0.75809	28	0.75033	10	0.78896	8	0.77344
	8	364	0.75085	153	0.75042	55	0.76287	20	0.75453	15	0.78802
	10	459	0.75003	192	0.75345	69	0.76475	24	0.80039	19	0.75966
0.75	0	73	0.25409	31	0.25015	11	0.25921	3	0.39432	2	0.50000
	2	208	0.25009	86	0.25629	31	0.25124	10	0.28800	7	0.34375
	4	332	0.25173	138	0.25484	49	0.25861	16	0.28817	12	0.27441
	8	572	0.25127	238	0.25392	85	0.25124	28	0.27323	21	0.25172
	10	690	0.25038	287	0.25361	102	0.25631	34	0.26439	25	0.27063
0.95	0	158	0.05041	65	0.05202	23	0.05129	7	0.06131	5	0.06250
	2	332	0.05060	138	0.05053	48	0.05331	15	0.06113	11	0.05469
	4	484	0.05003	201	0.05027	70	0.05361	22	0.06285	16	0.05923
	8	763	0.05028	317	0.05063	111	0.05347	36	0.05344	26	0.05388
	10	897	0.05019	373	0.05029	131	0.05236	42	0.05923	30	0.06802

Table 2. SASP for SHCJ distribution with parameter: $\gamma = 0.30$ for different values of θ

α^*	c	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\theta = 0.25$											
0.25	0	16	0.75006	6	0.78253	3	0.75681	1	1	1	1
	2	92	0.75047	37	0.75302	14	0.76607	5	0.85029	4	0.87500
	4	178	0.75248	71	0.75639	27	0.75625	10	0.78681	8	0.77344
	8	361	0.75155	144	0.75285	54	0.75340	20	0.75120	15	0.78802
	10	455	0.75127	181	0.75443	68	0.75014	24	0.79714	19	0.75966
0.75	0	73	0.25146	29	0.25328	10	0.28539	3	0.39289	3	0.25000
	2	206	0.25139	81	0.25714	30	0.25347	10	0.28556	7	0.34375
	4	330	0.25059	130	0.25571	48	0.25138	16	0.28505	12	0.27441
	8	568	0.25075	225	0.25133	82	0.25825	28	0.26917	21	0.25172
	10	685	0.25010	271	0.25212	99	0.25675	34	0.25999	25	0.27063
0.95	0	157	0.05023	62	0.05020	22	0.05362	7	0.06065	5	0.06250
	2	330	0.05025	130	0.05068	47	0.05107	15	0.06011	11	0.05469
	4	480	0.05024	189	0.05100	68	0.05328	22	0.06156	16	0.05923
	8	757	0.05041	299	0.05043	108	0.05246	36	0.05198	26	0.05388
	10	890	0.05031	351	0.05103	127	0.05283	42	0.05751	30	0.06802
$\theta = 0.30$											
0.25	0	15	0.75302	6	0.76652	2	0.86436	1	1	1	1
	2	87	0.75137	34	0.75697	13	0.78407	5	0.84814	4	0.87500
	4	169	0.75099	66	0.75361	26	0.75476	10	0.78293	8	0.77344
	8	342	0.75124	133	0.75418	52	0.75051	19	0.80364	15	0.78802
	10	431	0.75101	168	0.75085	65	0.75457	24	0.79122	19	0.75966
0.75	0	69	0.25213	27	0.25091	10	0.26931	3	0.39032	3	0.25000
	2	195	0.25152	75	0.25596	28	0.27111	10	0.28122	7	0.34375
	4	312	0.25167	121	0.25002	46	0.25163	16	0.27950	12	0.27441
	8	538	0.25030	208	0.25082	79	0.25275	28	0.26197	21	0.25172
	10	648	0.25102	250	0.25396	95	0.25545	34	0.25218	25	0.27063
0.95	0	148	0.05087	57	0.05089	21	0.05418	7	0.05947	5	0.06250
	2	312	0.05054	120	0.05077	45	0.05113	15	0.05830	11	0.05469
	4	454	0.05048	175	0.05023	65	0.05390	22	0.05930	16	0.05923
	8	717	0.05021	276	0.05062	104	0.05039	35	0.06250	26	0.05388
	10	843	0.05008	324	0.05125	122	0.05162	42	0.05451	30	0.06802

Table 3. SASP for SHCJ distribution with parameter: $\gamma = 0.40$ for different values of θ

α^*	c	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\theta = 0.15$											
0.25	0	21	0.75780	8	0.77017	3	0.78235	1	1	1	1
	2	126	0.75209	48	0.75306	16	0.75353	5	0.85575	4	0.87500
	4	245	0.75239	93	0.75214	30	0.76140	10	0.79662	8	0.77344
	8	498	0.75020	188	0.75147	60	0.76262	20	0.76639	15	0.78802
	10	627	0.75099	237	0.75036	76	0.75494	25	0.76274	19	0.75966
0.75	0	100	0.25340	38	0.25149	12	0.25924	4	0.25252	3	0.25000
	2	284	0.25153	107	0.25052	33	0.26859	10	0.29689	7	0.34375
	4	455	0.25082	171	0.25089	54	0.25248	16	0.29960	12	0.27441
	8	784	0.25004	294	0.25202	93	0.25115	28	0.28816	21	0.25172
	10	944	0.25110	355	0.25005	112	0.25195	34	0.28064	25	0.27063
0.95	0	217	0.05003	81	0.05056	25	0.05258	7	0.06377	5	0.06250
	2	456	0.05001	170	0.05105	53	0.05144	15	0.06496	11	0.05469
	4	663	0.05005	248	0.05042	77	0.05240	23	0.05114	16	0.05923
	8	1045	0.05034	391	0.05088	122	0.05216	36	0.05903	26	0.05388
	10	1229	0.05008	460	0.05060	144	0.05087	43	0.05332	30	0.06802
$\theta = 0.15$											
0.25	0	19	0.75297	7	0.77598	3	0.77161	1	1	1	1
	2	111	0.75245	42	0.75978	15	0.76235	5	0.85363	4	0.87500
	4	216	0.75208	82	0.75554	29	0.75108	10	0.79282	8	0.77344
	8	438	0.75161	166	0.75426	57	0.76390	20	0.76051	15	0.78802
	10	552	0.75154	209	0.75457	72	0.75902	25	0.75610	19	0.75966
0.75	0	88	0.25377	33	0.25855	11	0.27352	4	0.25007	2	0.50000
	2	250	0.25173	94	0.25489	32	0.25525	10	0.29244	7	0.34375
	4	401	0.25012	151	0.25232	51	0.25705	16	0.29388	12	0.27441
	8	690	0.25049	260	0.25231	88	0.25503	28	0.28066	21	0.25172
	10	831	0.25135	314	0.25015	106	0.25598	34	0.27248	25	0.27063
0.95	0	191	0.05004	71	0.05187	24	0.05071	7	0.06253	5	0.06250
	2	401	0.05024	150	0.05148	50	0.05283	15	0.06303	11	0.05469
	4	583	0.05035	219	0.05074	73	0.05265	22	0.06525	16	0.05923
	8	920	0.05030	346	0.05054	116	0.05139	36	0.05619	26	0.05388
	10	1082	0.05004	407	0.05031	136	0.05291	43	0.05046	30	0.06802

Table 4. SASP for SHCJ distribution with parameter: $\gamma = 0.50$ for different values of θ

α^*	c	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.8$		$\alpha = 1$	
		n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
$\theta = 0.25$											
0.25	0	20	0.76068	7	0.76863	3	0.76703	1	1	1	1
	2	121	0.75374	41	0.75460	15	0.75217	5	0.85296	4	0.87500
	4	236	0.75263	79	0.75668	28	0.76129	10	0.79161	8	0.77344
	8	479	0.75174	160	0.75522	56	0.76121	20	0.75865	15	0.78803
	10	604	0.75121	202	0.75287	71	0.75196	25	0.75399	19	0.75966
0.75	0	97	0.25105	32	0.25676	11	0.26550	3	0.39611	3	0.25000
	2	274	0.25069	91	0.25266	31	0.26205	10	0.29104	7	0.34375
	4	438	0.25145	146	0.25041	50	0.25581	16	0.29209	12	0.27441
	8	755	0.25047	251	0.25137	86	0.25680	28	0.27832	21	0.25172
	10	910	0.25044	302	0.25337	104	0.25322	34	0.26993	25	0.27063
0.95	0	209	0.05006	69	0.05067	23	0.05407	7	0.06215	5	0.06250
	2	439	0.05016	145	0.05090	49	0.05244	15	0.06243	11	0.05469
	4	638	0.05036	211	0.05099	72	0.05025	22	0.06449	16	0.05923
	8	1007	0.05023	334	0.05016	113	0.05316	36	0.05532	26	0.05388
	10	1184	0.05007	392	0.05081	134	0.05000	42	0.06144	30	0.06802
$\theta = 0.30$											
0.25	0	22	0.75128	7	0.75670	3	0.75938	1	1	1	1
	2	128	0.75305	39	0.75232	14	0.77137	5	0.85134	4	0.87500
	4	250	0.75093	75	0.75446	27	0.76365	10	0.78869	8	0.77344
	8	507	0.75020	152	0.75118	54	0.76379	20	0.75412	15	0.78802
	10	638	0.75152	191	0.75294	68	0.76182	24	0.79999	19	0.75966
0.75	0	102	0.25274	30	0.25990	11	0.25252	3	0.39414	3	0.25000
	2	289	0.25185	86	0.25273	30	0.26086	10	0.28769	7	0.34375
	4	463	0.25125	138	0.25031	48	0.26080	16	0.28779	12	0.27441
	8	798	0.25033	237	0.25228	83	0.25753	28	0.27272	21	0.25172
	10	962	0.25009	286	0.25099	100	0.25845	34	0.26384	25	0.27063
0.95	0	220	0.05068	65	0.05111	22	0.05557	7	0.06123	5	0.06250
	2	464	0.05016	137	0.05087	47	0.05398	15	0.06100	11	0.05469
	4	675	0.05007	200	0.05002	69	0.05238	22	0.06269	16	0.05923
	8	1064	0.05034	315	0.05081	109	0.05342	36	0.05326	26	0.05388
	10	1251	0.05019	371	0.05011	129	0.05112	42	0.05902	30	0.06802

5 Classical Estimation Procedures

In this section, we discuss some classical methods for the estimation the parameters of the proposed SHCJ distribution.

Definition 5.1 (Maximum likelihood estimation (MLE)). Let $(x_1, x_2, ..., x_n)$ be random samples of size n drawn from the SHCJ distribution, then the likelihood function is given by

$$\begin{aligned}\ell(f_{SCJ}(x,\theta,\gamma) &= \prod_{i=1}^{n} \frac{\theta^2}{\theta+2}(1+\theta(\theta(x-\gamma)^2)e^{-}(\theta(x-\gamma) \\ &= \frac{\theta^{2n}}{(\theta+2)^n} e^{-\theta\sum(x-\gamma)} \prod_{i=1}^{n}(1+\theta(x-\gamma)^2)\end{aligned} \tag{36}$$

Taking the natural log of ℓ and differentiating w.r.t θ yields the following result.

$$\begin{aligned}\psi = \ell(x,\theta) &= 2n\ln\theta - n\ln(\theta+2) - \theta\sum(x-\gamma) + \sum_{x=1}^{n}\ln(1+\theta(x-\gamma)^2) \\ \frac{\partial\theta}{\partial\psi} &= \frac{2n}{\theta} - \frac{n}{\theta+2} - \sum(x_i-\gamma) + \sum_{i-1}^{n}\left(\frac{(x_i-\gamma)^2}{1+\theta(x_i-\gamma)^2}\right)\end{aligned} \tag{37}$$

equating to zero

$$\frac{2n}{\theta} - \frac{n}{\theta+2} - \sum(x_i-\gamma) + \sum_{i=1}^{n}\left(\frac{(x_i-\gamma)^2}{1+\theta(x_i-\gamma)^2}\right) = 0$$

We consider the estimated value of γ equal to the minimum value of the random variable X in case the maximum likelihood approach for estimation is used in this lifetime SHCJ distribution, where there is a relation between the random variable X and the parameter γ where $(x > \gamma)$. We estimate the other parameter θ, using the projected value of γ as the minimum value of X. Since there is no closed-form solution for equations (30) and (31), estimations of λ and k are determined iteratively using the Newton-Raphson method Albert [25] and Gelman [26].

Definition 5.2 (Maximum product space estimators (MPSE)). The maximum product spacing method, which approaches the Kullback-Leibler information measure, is an acceptable substitute for the highest likelihood strategy. Assume for a moment that the data are now arranged in ascending order. Following that, the SHCJ's maximum product spacing is provided as follows.

$$Gs(\theta,\gamma|data) = \left(\prod_{i=1}^{n+1} D_l(x_i\theta,\gamma)\right)^{\frac{1}{n+1}}, \tag{38}$$

where $D_l(x_i,\theta,\gamma) = F(x_i;\theta,\gamma) - F(x_{i-1};\theta,\gamma)$, $i = 1,2,3,...,n$

In a similar manner, one may decide to increase the function.

$$H(\theta,\gamma) = \frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(\theta,\gamma). \tag{39}$$

The parameter estimates are determined by calculating the first derivative of the function $H(\vartheta)$ with respect to θ, and γ, solving the resulting nonlinear equations at $\frac{\partial H(\phi)}{\partial\theta} = 0$, and $\frac{\partial H(\phi)}{\partial\gamma} = 0$, where $\phi = (\theta,\gamma)$,

Definition 5.3 (Least squares estimation (LSE))**.** Swain et al [27] suggested using the least squares estimation to estimate the Beta distribution's parameters. Using the inferences from the study of Swain et al. [27], we write

$$E[F(x_{i:n}|\theta,\gamma)] = \frac{i}{n+1}.$$
$$V[F(x_{i:n}|\theta,\gamma)] = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

The least squares estimates $\hat{\theta}_{LSE}$ and $\hat{\gamma}_{LSE}$ of the parameter θ and γ are obtained by minimizing the function $L(\theta,\gamma)$ with respect to θ and γ

$$L(\theta,\gamma) = \underset{(\theta,\gamma)}{\arg\min} \sum_{i=1}^{n} \Big[F(x_{i:n}|\theta,\gamma) - \frac{i}{n+1}\Big]^2. \tag{40}$$

The estimates are obtained by solving the following non-linear equations

$$\begin{aligned} &\sum_{i=1}^{n} \Big[F(x_{i:n}|\theta,\gamma) - \frac{i}{n+1}\Big]^2 \Delta_1(x_{i:n}|\theta,\gamma) = 0 \\ &\sum_{i=1}^{n} \Big[F(x_{i:n}|\theta,\gamma) - \frac{i}{n+1}\Big]^2 \Delta_2(x_{i:n}|\theta,\gamma) = 0 \end{aligned} \tag{41}$$

where

$$\begin{aligned} \Delta_1(x_{i:n}|\theta,\gamma) &= \left\{ \frac{1}{x-\gamma} + \frac{\theta^2(x-\gamma)+2\theta}{\theta+2} - \frac{(\theta+2)\left[(x-\gamma)\left(\theta(x-\gamma)+2\right)+\theta(x-\gamma)^2\right] - \theta^2(x-\gamma)^2 + 2\theta(x-\gamma)}{(\theta+2)^2} \right\} e^{-\theta(x-\gamma)} \\ \Delta_2(x_{i:n}|\theta,\gamma) &= \frac{1}{\theta+2}\left(2\theta^2 x - 2\theta^2\gamma + \theta - 2 - \theta^2 x^2 + 2\theta^2 x^2 + 2x\theta^2\gamma - \gamma^2 - 2\theta x + 2\theta\gamma\right) e^{-\theta(x-\gamma)} \end{aligned} \tag{42}$$

Definition 5.4 (Weighted least squares estimation (WLSE))**.** The weighted least squares estimates $\hat{\theta}_{WLSE}$ and $\hat{\gamma}_{WLSE}$ of the SHCJ distribution parameters θ and γ are obtained by minimizing the function $W(\theta,\gamma)$ with respect to θ and γ

$$W(\theta,\gamma) = \underset{(\theta,\gamma)}{\arg\min} \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta,\gamma) - \frac{i}{n+1}\right]^2. \tag{43}$$

Solving the following non-linear equations yields the estimates

$$\begin{aligned} &\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta,\gamma) - \frac{i}{n+1}\right] \Delta_1(x_{i:n}|\theta,\gamma) = 0, \\ &\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta,\gamma) - \frac{i}{n+1}\right] \Delta_2(x_{i:n}|\theta,\gamma) = 0 \end{aligned} \tag{44}$$

where $\Delta_1(x.|\theta,\gamma)$ and $\Delta_2(.|\theta,\gamma)$ are as defined in (35).

Definition 5.5 (Cramér-von-Mises estimation (CVME))**.** The Cramér-von-Mises estimates $\hat{\theta}_{CVME}$ and $\hat{\gamma}_{CVME}$ of the SHCJ distribution parameters θ and γ are obtained by minimizing the function $C(\theta,\gamma)$ with respect to θ and γ

$$C(\theta,\gamma) = \underset{(\theta,\gamma)}{\arg\min} \left\{ \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{i:n}|\theta,\gamma) - \frac{2i-1}{2n}\right]^2 \right\}. \tag{45}$$

The estimates are obtained by solving the following non-linear equations

$$\begin{aligned}\sum_{i=1}^{n}\left(F(x_{i:n}|\theta,\gamma)-\frac{2i-1}{2n}\right)\Delta_1(x_{i:n}|\theta,\gamma)=0\\ \sum_{i=1}^{n}\left(F(x_{i:n}|\theta,\gamma)-\frac{2i-1}{2n}\right)\Delta_2(x_{i:n}|\theta,\gamma)=0\end{aligned}\tag{46}$$

where $\Delta_1(.|\theta,\gamma)$ and $\Delta_2(.|\theta,\gamma)$ are as defined in (35).

Definition 5.6 (Anderson-Darling estimation (ADE)). The Anderson-Darling estimates $\hat{\theta}_{ADE}$ and $\hat{\gamma}_{ADE}$ of the SHCJ distribution parameters θ and γ are obtained by minimizing the function $A(\theta,\gamma)$ with respect to θ and γ

$$A(\theta,\gamma)=\underset{(\theta,\gamma)}{\arg\min}\sum_{i=1}^{n}(2i-1)\Big\{\ln F(x_{i:n}|\theta,\gamma)+\ln\Big[1-F(x_{n+1-i:n}|\theta,\gamma)\Big]\Big\}.\tag{47}$$

The estimates are obtained by solving the following sets of non-linear equations

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{i:n}|\theta,\gamma)}{F(x_{i:n}|\theta,\gamma)}-\frac{\Delta_1(x_{n+1-i:n}|\theta,\gamma)}{1-F(x_{n+1-i:n}|\theta,\gamma)}\right]=0\\ \sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{i:n}|\theta,\gamma)}{F(x_{i:n}|\theta,\gamma)}-\frac{\Delta_2(x_{n+1-i:n}|\theta,\gamma)}{1-F(x_{n+1-i:n}|\theta,\gamma)}\right]=0\end{aligned}\tag{48}$$

where $\Delta_1(.|\theta,\gamma)$ and $\Delta_2(.|\theta,\gamma)$ are as defined in (35).

Definition 5.7 (Right-Tailed Anderson-Darling estimation (RTADE)). The Right-Tailed Anderson-Darling estimates $\hat{\theta}_{RTADE}$ and $\hat{\gamma}_{RTADE}$ of the SHCJ distribution parameters θ and γ are obtained by minimizing the function $R(\theta,\gamma)$ with respect to θ and γ

$$R(\theta,\gamma)=\underset{(\theta,\gamma)}{\arg\min}\left\{\frac{n}{2}-2\sum_{i=1}^{n}F(x_{i:n}|\theta,\gamma)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\ln\Big[1-F(x_{n+1-i:n}|\theta,\gamma)\Big]\right\}.\tag{49}$$

The following set of non-linear equations can be solved to obtain the estimates.

$$\begin{aligned}-2\sum_{i=1}^{n}\frac{\Delta_1(x_{i:n}|\theta,\gamma)}{F(x_{i:n}|\theta,\gamma)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{n+1-i:n}|\theta,\gamma)}{1-F(x_{n+1-i:n}|\theta,\gamma)}\right]=0\\ -2\sum_{i=1}^{n}\frac{\Delta_2(x_{i:n}|\theta,\gamma)}{F(x_{i:n}|\theta,\gamma)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{n+1-i:n}|\theta,\gamma)}{1-F(x_{n+1-i:n}|\theta,\gamma)}\right]=0\end{aligned}\tag{50}$$

where $\Delta_1(.|\theta,\gamma)$ and $\Delta_2(.|\theta,\gamma)$ are as defined in (35). The estimates given in (31), (32), (34), (37), (39), (41), (43) are obtained using **optim()** function in R with the Newton-Raphson iterative algorithm.

6 Bayesian Estimation of SHCJ Distribution Parameters

This section deals with the Bayesian estimation (BE) of the unknown parameters of the SHCJ distribution. For Bayesian parameter estimation, many loss functions, including squared error, LINEX, and generalized entropy loss functions, can be taken into consideration by Albert [25] and Mood [28]. We can consider applying independent gamma priors for the variables θ and γ with pdfs in the parameter prior distributions of SHCJ.

$$\begin{aligned}\pi_1(\lambda)\propto\theta^{s_1-1}\mathrm{e}^{-q_1\theta}\quad&\theta>0,s_1>0,q_1>0,\\ \pi_2(k)\propto\gamma^{s_2-1}\mathrm{e}^{-q_2\gamma}\quad&\gamma>0,s_2>0,q_2>0\end{aligned}\tag{51}$$

where the hyper-parameters $s_j, q_j, j = 1, 2$ are selected to reflect the prior knowledge about the unknown parameters. The joint prior for $\phi = (\theta, \gamma)$ is given by

$$\begin{aligned} \pi(\phi) &= \pi_1(\theta)\pi_2(\gamma) \\ \pi(\phi) &\propto \theta^{s_1-1}\gamma^{s_2-1}\mathrm{e}^{\{-q_1\theta-q_2\gamma\}}. \end{aligned} \tag{52}$$

The corresponding posterior density given the observed data $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is given by:

$$\pi(\phi \mid \mathbf{x}) = \frac{\pi(\phi)\ell(\phi)}{\int_\phi \pi(\phi)\ell(\phi)d\phi}, \tag{53}$$

consequently, the posterior density function is denoted by:

$$\pi(\phi \mid \mathbf{x}) \quad \propto \quad \theta^{2n+s_1-1}\gamma^{s_2-1}\left(\frac{\theta^2}{\theta+2}\right)^n e^{-q_1\theta-q_2\gamma+n\theta\gamma-\theta\sum x}\prod_{i=1}^{n}\left(1+\theta(x-\gamma)^2\right) \tag{54}$$

Given any function, such as $l(\phi)$ under the squared error loss (SEL) function, the Bayes estimator is given by

$$\hat{\phi}_{BE_{SEL}} = E\left[l(\phi)|\mathbf{x}\right] = \int_\phi l(\phi)\pi(\phi|x)d\phi. \tag{55}$$

The SEL impacts underestimation and overestimation equally because it has an asymmetric loss function. In many actual situations, both underestimation and overestimation can have serious implications. A proposed LINEX loss can be made in certain instances as an alternative to the SE loss given by

$$\left(l(\phi), \hat{l}(\phi)\right) = \mathrm{e}^{\left\{\hat{l}(\phi)-l(\phi)\right\}} - v\left(\hat{l}(\phi) - l(\phi)\right) - 1.$$

where $v \neq 0$ is a shape parameter. Here $v > 1$ suggests that an overestimation is more serious than an underestimation, and vice versa for $v < 0$. Further v approaching zero replicates the SE loss function itself. One may refer to Varian [29] and Doostparast et al. [30] for more details in this regard. The BE of $l(\phi)$ under this loss can be derived as

$$\hat{\phi}_{BE_{LINEX}} = E\left[\mathrm{e}^{\{-vl(\phi)\}}|\mathbf{x}\right] = -\frac{1}{v}\log\left[\int_\phi \mathrm{e}^{\{-vl(\phi)\}}\pi(\phi|x)d\phi\right]. \tag{56}$$

Additionally, we take into account the general entropy loss (GEL) function suggested by Calabria and Pulcini [31], which is defined as follows.

$$\left(l(\phi), \hat{l}(\phi)\right) = \left(\frac{\hat{l}(\phi)}{l(\phi)}\right)^\tau - \tau\log\left(\frac{\hat{l}(\phi)}{l(\phi)}\right) - 1,$$

where the shape parameter $\tau \neq 0$ denotes a departure from symmetry. It views overestimation as more significant than underestimating when $\tau > 0$ and the opposite is true when $\tau < 0$. Given is the Bayes estimator with regard to the GE loss function.

$$\hat{\phi}_{BE_{GEL}} = \left[E\left((l(\phi))^{-\tau}|\mathbf{x}\right)\right]^{-1/\tau} = \left[\int_\phi (l(\phi))^{-\tau}\pi(\phi|x)d\phi\right]^{-1/\tau}. \tag{57}$$

The estimations produced by (54), (55), and (56) can be seen to not be able to be transformed into closed-form expressions. We then use the Markov chain Monte Carlo (MCMC) approach to generate posterior samples and arrive at suitable BEs.

6.1 Markov chain Monte Carlo

A general simulation technique for computing posterior quantities of interest and sampling from posterior distributions is the MCMC technique. Read Ravenzwaaij *et al.* [32] and Albert [25] for further details on MCMC. In fact, using a kernel estimate of the posterior distribution and the MCMC samples, it is possible to properly quantify the posterior uncertainty with regard to the parameter ϕ.

As a stochastic process, a Markov chain

$$\phi^{(0)}, \phi^{(1)}, \phi^{(2)}, \ldots$$

.

is a random variable $\phi^{(i)}$ with values that are contained in a "state space." The state space, or the process's state at time i, remains constant across time i. The following Markov property applies to Markov chains: Only via the current state $\phi^{(i)}$ does the distribution of the next state $\phi^(i+1)$ depend on the past $\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(i)}$. The homogeneous Markov chains employed in MCMC approaches allow the conditional distribution of $\phi^{(i+1)}$ given $\phi^{(i)}$ to be independent of the index i. MCMC sample selection from a distribution:

1. Starting with an initial guess: simply one possible value that could be taken as coming from the distribution.
2. creating a number of fresh samples from this first hunch. Two steps are used to create each new sample:
   - A modest random perturbation is added to the most recent sample to create a proposal for the new sample.
   - Acceptance: Whether the new suggestion is rejected or accepted as the new sample (in which case the old sample is retained).

There are various techniques for integrating randomness to produce proposals, as well as numerous methods for acceptance and rejection, including Gibbs sampling and the Metropolis-Hastings algorithm.

6.2 Metropolis-Hasting algorithm

The algorithm requires that the proposed distribution and the initial values of the unknown parameters ϕ be defined. A multivariate normal distribution, defined as $q\left(\phi^{'}|\phi\right) \equiv N_2\big(\phi, S_\phi\big)$, will be taken into account for the proposal distribution, where S_ϕ denotes the variance-covariance matrix. It's possible to make unfavorable observations, which is unacceptable. The MLEs for ϕ are taken into account for the starting values, i.e., $\phi^{(0)} = \hat{\phi}_{MLE}$. The Fisher information matrix $I(.)$ is used to pick S_ϕ, which is thought to be the asymptotic variance-covariance matrix $I^{-1}\left(\hat{\phi}_{MLE}\right)$. It is noted that the choice of S_ϕ, which affects the MH algorithm's acceptance rate, is a critical decision.

In this regard, the MH algorithm's procedures for taking a sample from the provided posterior density (referred to as "(53)") are as follows:

Step 1. Set initial value of ϕ as $\phi^{(0)} = \left(\hat{\theta}_{MLE}, \hat{\gamma}_{MLE}\right)$.

Step 2. For $i = 1, 2, \ldots, M$ repeat the following steps:
[label=2.0:]Set $\phi = \phi^{(i-1)}$. Generate a new candidate parameter value δ from $N_2\left(\log\phi, S_\phi\right)$. Set $\theta^{'} = \exp(\delta)$. Calculate $\beta = \frac{\pi(\phi^{'}|x)}{\pi(\phi|x)}$, where $\pi(\cdot)$ is the posterior density in (53). Generate a sample u from the uniform $U(0,1)$ distribution. Accept or reject the new candidate $\theta^{'}$

$$\begin{cases} \text{If} \quad u \le \beta \quad \text{set} \quad \phi^{(i)} = \phi^{'} \\ \\ \text{otherwise} \quad \text{set} \quad \phi^{(i)} = \phi. \end{cases}$$

Finally, part of the initial samples can be eliminated (burn-in) from the random samples of size M derived from the posterior density, and the remaining samples can then be used to calculate Bayes estimates. Using MCMC under the SEL, LINEX, and GEL functions, the BEs of $\phi^{(i)} = \left(\theta^{(i)}, \gamma^{(i)}\right)$ can be calculated as follows.

$$\hat{\phi}_{BE_{SEL}} = \frac{1}{M - l_B} \sum_{i=l_B}^{M} \phi^{(i)}, \tag{58}$$

$$\hat{\phi}_{BE_{LINEX}} = -\frac{1}{v} \log \left[\frac{1}{M - l_B} \sum_{i=l_B}^{M} \mathrm{e}^{\left\{-v\phi^{(i)}\right\}} \right], \tag{59}$$

$$\hat{\phi}_{BE_{GEL}} = \left[\frac{1}{M - l_B} \sum_{i=l_B}^{M} \left(\phi^{(i)}\right)^{-\tau} \right]^{-1/\tau}, \tag{60}$$

where l_B represents the number of burn-in samples.

7 Simulation Study

In order to evaluate the efficacy of the estimating methods (Non-BEs and BEs) outlined in the previous part, we simulate data for the SHCJ in this subsection. From the SHCJ distribution, we produce 1000 data by using the initial parameter values as

- $\gamma = 0.10$ and $\theta = 0.25$
- $\gamma = 0.20$ and $\theta = 0.25$
- $\gamma = 0.10$ and $\theta = 0.30$
- $\gamma = 0.15$ and $\theta = 0.30$

and sample sizes $n = 25, 50, 75, 100$. For each estimate $\hat{\phi} = (\hat{\lambda}, \hat{\theta})$, we compute the Bias and Root Mean Squared Error(RMSE) respectively as

$$Bias(\hat{\phi}) = \frac{1}{B} \sum_{i=1}^{B} (\hat{\phi}_i - \phi),$$

and

$$RMSE(\hat{\phi}) = \sqrt{\frac{1}{B} \sum_{i=1}^{B} (\hat{\phi}_i - \phi)^2}.$$

We used the Newton-Raphson algorithm to find the desired estimates for the Non-Bayesian procedure. For the Bayesian method, BEs are generated using MCMC and the MH algorithm with an informative prior. We made the assumption that all gamma distribution hyperparameters are equal to twice the parameter values while calculating the informative prior. The desired estimations are then calculated using these data as inputs. The MH algorithm is applied by the MLEs while taking into account initial estimate values. Out of the total 10,000 samples created from the posterior density and subsequently obtained BEs under various loss functions, SEL, LINEX at $v = -1.5, 1.5$, and finally GEL at $\tau = -0.5, 0.5, 2000$ burn-in samples are ultimately deleted. We calculate the bias and RMSE for each strategy.

Table 5. Average estimated Biases and RMSEs of different estimation methods for SHCJ distribution at different sample sizes n and different values of the parameters
$(\theta = 0.10, \quad \gamma = 0.25)$

Method	Para	$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	θ	0.01913	0.00076	0.01121	0.00029	0.00811	0.00015	0.00624	0.00011
	γ	4.71567	31.24686	3.06779	14.26182	2.25224	8.05646	1.85936	5.79146
MPSE	θ	0.00538	0.00028	0.00409	0.00013	0.00292	0.00008	0.00264	0.00006
	γ	1.71003	14.90960	1.10806	6.01073	0.78445	3.54608	0.61525	2.49367
LSE	θ	0.00066	0.00039	0.00095	0.00019	0.00033	0.00012	0.00068	0.00009
	γ	0.85510	20.97618	0.54673	9.76933	0.29860	6.60037	0.27739	5.13413
WLSE	θ	0.00004	0.00034	0.00039	0.00015	0.00007	0.00009	0.00023	0.00007
	γ	0.55143	15.71176	0.31275	6.66316	0.13390	4.47784	0.10829	3.41891
CVME	θ	0.00512	0.00045	0.00197	0.00020	0.00159	0.00013	0.00082	0.00009
	γ	0.62914	18.50024	0.21140	9.06661	0.19025	6.35260	0.11000	5.01953
ADE	θ	0.00091	0.00030	0.00060	0.00014	0.00042	0.00008	0.00088	0.00006
	γ	0.21375	12.84476	0.32596	5.73764	0.23992	3.74012	0.26312	2.84236
RTADE	θ	0.00356	0.00044	0.00110	0.00018	0.00114	0.00012	0.00037	0.00009
	γ	0.30370	23.55565	0.06677	11.08676	0.10957	8.01270	0.01808	6.37554
BE_{Linex1}	θ	0.53391	0.65604	0.60812	0.73497	0.70768	0.88735	0.74518	0.92710
	γ	2.69701	8.71887	2.95949	9.82915	3.14843	10.88152	3.38241	12.24370
BE_{Linex2}	θ	0.53685	0.66487	0.60856	0.73796	0.70692	0.88843	0.73981	0.91749
	γ	2.74312	9.05196	3.01835	10.19951	3.19589	11.16133	3.41211	12.41691
BE_{GEL1}	θ	0.53722	0.66072	0.61206	0.74161	0.71083	0.89151	0.74362	0.92065
	γ	2.67040	8.55175	3.00613	10.17771	3.14123	10.87274	3.33181	11.92303
BE_{GEL2}	θ	0.53467	0.65864	0.61004	0.73989	0.70830	0.88876	0.74266	0.92131
	γ	2.67644	8.61058	2.98341	10.03886	3.13127	10.79717	3.34824	12.01957
BE_{SEL}	θ	0.52975	0.64989	0.60459	0.72884	0.70131	0.87303	0.73895	0.91306
	γ	2.65161	8.50968	2.96629	9.98800	3.11117	10.72435	3.31052	11.80404

Table 6. Average estimated Biases and RMSEs of different estimation methods for SHCJ distribution at different sample sizes n and different values of the parameters ($\theta = 0.20, \quad \gamma = 0.25$)

Method	Para	$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	θ	0.02778	0.00195	0.01430	0.00065	0.00956	0.00038	0.00804	0.00027
	γ	1.76021	5.09488	1.02160	1.84479	0.69759	0.90765	0.58422	0.65188
MPSE	θ	0.01101	0.00105	0.00701	0.00045	0.00480	0.00031	0.00269	0.00021
	γ	0.86078	3.54995	0.42524	1.28499	0.27124	0.64492	0.12157	0.38275
LSE	θ	0.00394	0.00143	0.00186	0.00067	0.00189	0.00048	0.00098	0.00036
	γ	0.61679	5.30440	0.26204	2.65058	0.23378	1.78447	0.15076	1.20132
WLSE	θ	0.00261	0.00117	0.00136	0.00050	0.00133	0.00035	0.00057	0.00025
	γ	0.44226	3.64136	0.18062	1.67315	0.15560	1.03428	0.09336	0.68258
CVME	θ	0.00735	0.00160	0.00387	0.00072	0.00196	0.00049	0.00203	0.00037
	γ	0.12822	4.36878	0.10884	2.46774	0.01720	1.65988	0.04390	1.15453
ADE	θ	0.00011	0.00111	0.00177	0.00047	0.00259	0.00032	0.00186	0.00023
	γ	0.24893	3.17274	0.19556	1.49475	0.23277	0.92211	0.17145	0.61787
RTADE	θ	0.00360	0.00148	0.00190	0.00067	0.00071	0.00050	0.00102	0.00037
	γ	0.08251	5.72221	0.00779	3.05388	0.06033	2.20422	0.01308	1.48429
BE_{Linex1}	θ	2.17452	5.55561	2.24933	5.66734	2.30655	5.90955	2.32566	5.99775
	γ	1.53524	3.45435	1.58015	2.89952	0.28009	0.07845	0.25561	0.06534
BE_{Linex2}	θ	2.20254	5.70653	2.24938	5.65660	2.32390	5.99444	2.32979	6.00863
	γ	1.50607	3.23983	1.61737	3.03565	0.28417	0.08075	0.25677	0.06593
BE_{GEL1}	θ	2.15630	5.46271	2.22511	5.54495	2.29154	5.84030	2.31723	5.96218
	γ	1.50709	3.25068	1.54533	2.77603	2.12929	7.96775	0.25446	0.06475
BE_{GEL2}	θ	2.16483	5.51338	2.23239	5.58449	2.30087	5.89077	2.32594	6.01259
	γ	1.51311	3.35976	1.56067	2.83911	0.27325	0.07467	0.25332	0.06417
BE_{SEL}	θ	2.14545	5.42982	2.21075	5.48953	2.27597	5.77733	2.30402	5.91362
	γ	1.50139	3.28984	1.52282	2.72425	0.26076	0.06800	0.24873	0.06187

Table 7. Average estimated Biases and RMSEs of different estimation methods for SHCJ distribution at different sample sizes n and different values of the parameters $(\theta = 0.10, \quad \gamma = 0.30)$

Method	Para	$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	θ	0.02089	0.00082	0.01058	0.00026	0.00863	0.00016	0.00656	0.00011
	γ	4.83155	32.59397	3.02927	14.09211	2.22677	7.83292	1.77893	5.27900
MPSE	θ	0.00422	0.00026	0.00449	0.00013	0.00227	0.00007	0.00205	0.00006
	γ	1.55322	14.08447	1.11484	6.38712	0.72442	3.31758	0.58494	2.44056
LSE	θ	0.00016	0.00035	0.00084	0.00019	0.00037	0.00011	0.00017	0.00009
	γ	0.84265	19.98712	0.47392	11.03100	0.20419	5.84732	0.17219	5.04800
WLSE	θ	0.00082	0.00031	0.00041	0.00015	0.00073	0.00009	0.00053	0.00007
	γ	0.48679	14.97477	0.24611	7.49046	0.06743	4.02141	0.03727	3.36097
CVME	θ	0.00558	0.00042	0.00209	0.00021	0.00233	0.00012	0.00166	0.00010
	γ	0.59645	18.10951	0.27748	10.36808	0.29054	5.79545	0.20347	4.97913
ADE	θ	0.00197	0.00029	0.00080	0.00014	0.00029	0.00008	0.00010	0.00006
	γ	0.10867	12.38163	0.31604	6.35535	0.16170	3.40120	0.18240	2.76108
RTADE	θ	0.00405	0.00039	0.00093	0.00020	0.00194	0.00012	0.00128	0.00010
	γ	0.30372	22.93026	0.02784	13.07086	0.21460	7.60805	0.12886	6.13226
BE_{Linex1}	θ	0.57299	0.71853	0.62256	0.75608	0.71065	0.87070	0.75721	0.95306
	γ	3.01493	11.09623	3.41555	13.29395	3.64072	14.71969	3.70733	14.87272
BE_{Linex2}	θ	0.57347	0.72241	0.62194	0.75717	0.70969	0.87040	0.75609	0.95341
	γ	3.09499	11.69708	3.51625	14.07338	3.67361	14.89656	3.82783	15.81491
BE_{GEL1}	θ	0.57543	0.72105	0.62305	0.75470	0.71546	0.87965	0.75820	0.95245
	γ	3.02828	11.23092	3.34500	12.76779	3.55392	14.03523	3.65956	14.57919
BE_{GEL2}	θ	0.56868	0.70990	0.61828	0.74695	0.70999	0.86982	0.75601	0.95043
	γ	2.97958	10.87210	3.39186	13.15237	3.61163	14.51912	3.67132	14.61390
BE_{SEL}	θ	0.56899	0.71224	0.61415	0.73873	0.71095	0.87298	0.75080	0.93870
	γ	2.98379	11.01295	3.34456	12.87068	3.54711	14.07036	3.60164	14.12653

Table 8. Average estimated Biases and RMSEs of different estimation methods for SHCJ distribution at different sample sizes n and different values of the parameters ($\theta = 0.15, \quad \gamma = 0.30$)

Method	Para	$n = 25$		$n = 50$		$n = 75$		$n = 100$	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
MLE	θ	0.02617	0.00149	0.01368	0.00047	0.00964	0.00026	0.00782	0.00018
	γ	2.78703	11.87201	1.62999	4.40073	1.10029	2.19598	0.95009	1.61458
MPSE	θ	0.00709	0.00061	0.00523	0.00026	0.00353	0.00017	0.00213	0.00012
	γ	1.09180	6.72834	0.65925	2.44747	0.49410	1.41185	0.24394	0.81737
LSE	θ	0.00111	0.00077	0.00181	0.00038	0.00003	0.00028	0.00017	0.00019
	γ	0.65345	9.07338	0.42526	4.58512	0.25774	2.97991	0.10217	1.99692
WLSE	θ	0.00016	0.00068	0.00078	0.00029	0.00025	0.00021	0.00051	0.00015
	γ	0.42476	6.72370	0.25081	2.95505	0.17943	1.88475	0.03599	1.26039
CVME	θ	0.00755	0.00091	0.00242	0.00040	0.00294	0.00030	0.00240	0.00020
	γ	0.32543	7.95153	0.06572	4.11686	0.07938	2.83699	0.15005	1.96516
ADE	θ	0.00189	0.00066	0.00108	0.00027	0.00045	0.00019	0.00030	0.00013
	γ	0.18873	5.88943	0.26366	2.53973	0.24788	1.62022	0.12293	1.06862
RTADE	θ	0.00480	0.00091	0.00186	0.00042	0.00214	0.00029	0.00196	0.00021
	γ	0.02835	10.80858	0.00989	5.51147	0.01115	3.51831	0.10930	2.58956
BE_{Linex1}	θ	1.38050	2.63199	1.58160	3.01536	1.64980	3.11906	1.70359	3.27510
	γ	2.01775	5.46138	2.07229	5.04493	2.06451	4.65058	2.42537	6.20856
BE_{Linex2}	θ	1.38450	2.65288	1.59982	3.08890	1.65471	3.13363	1.71516	3.31845
	γ	2.05030	5.62587	2.13403	5.34788	2.12484	4.95082	2.52566	6.75370
BE_{GEL1}	θ	1.37284	2.59841	1.57364	2.98454	1.64299	3.09642	1.70623	3.29149
	γ	2.09620	6.00752	2.21312	6.08314	2.20209	5.63049	2.56253	7.37614
BE_{GEL2}	θ	1.37415	2.61133	1.57664	3.00039	1.64067	3.08847	1.70442	3.28616
	γ	1.98790	5.30984	2.04746	4.94121	2.04191	4.54791	2.39439	6.05090
BE_{SEL}	θ	1.35770	2.55677	1.56037	2.94545	1.63043	3.06132	1.69249	3.25072
	γ	2.00269	5.53087	2.11163	5.57604	1.99678	4.34746	2.33512	5.75600

Table 9. Confidence Intervals for MLEs and Credible Intervals for the Bayesian Estimates using BE_{SEL}, BE_{Linex1}, BE_{Linex2}, BE_{GEL1} & BE_{GEL2}

Initial values	Lower MLE	Upper MLE	Lower BE_{Linex1}	Upper BE_{Linex1}	Lower BE_{Linex2}	Upper BE_{Linex2}	Lower BE_{GEL1}	Upper BE_{GEL1}	Lower BE_{GEL2}	Upper BE_{GEL2}	Lower BE_{SEL}	Upper BE_{SEL}
$\theta = 0.10$	0.08691	1.74813	1.66122	0.10061	1.75413	1.65352	0.10000	1.84613	1.74613	0.09785	1.80830	1.71045
$\gamma = 0.25$	0.81478	4.81286	3.99808	1.51087	4.95792	3.44705	1.67445	4.99416	3.31971	2.01376	4.97008	2.95632
$\theta = 0.20$	0.20804	3.69536	3.48732	1.19632	3.97883	2.78251	1.20988	3.95078	2.74091	1.28945	3.97122	2.68177
$\gamma = 0.25$	0.15188	3.47168	3.31980	0.98090	2.78897	1.80807	0.53009	0.53009	0.00000	0.50561	0.50561	0.00000
$\theta = 0.10$	0.08664	1.83908	1.75244	0.09790	1.76185	1.66395	0.09999	1.78789	1.68790	0.10193	1.82467	1.72274
$\gamma = 0.30$	0.85470	5.76376	4.90906	1.42316	5.70769	4.28453	2.01347	5.93929	3.92581	2.12210	5.96545	3.84335
$\theta = 0.15$	0.14701	2.82380	2.67679	0.15443	2.75941	2.60497	0.76804	2.96664	2.19860	0.88262	2.98620	2.10358
$\gamma = 0.30$	0.10696	4.43418	4.32723	1.04198	4.26100	3.21902	1.73168	3.59980	1.86813	1.73034	3.62032	1.88998

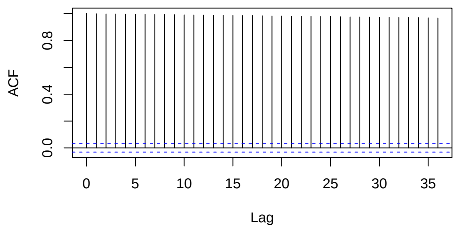

Fig. 5. Autocorrelation function plot for α

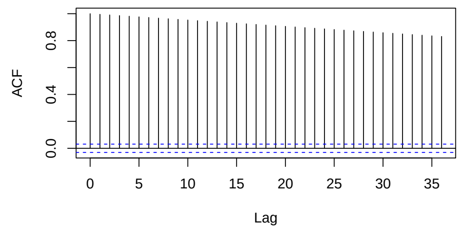

Fig. 6. Autocorrelation function plot for θ

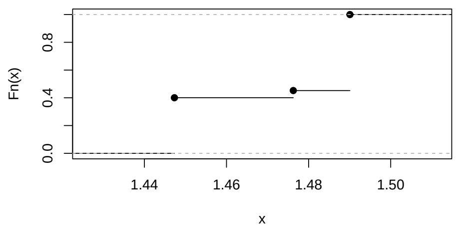

Fig. 7. Cumulative function plot for α

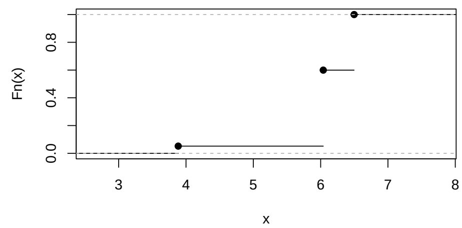

Fig. 8. Cumulative function plot for θ

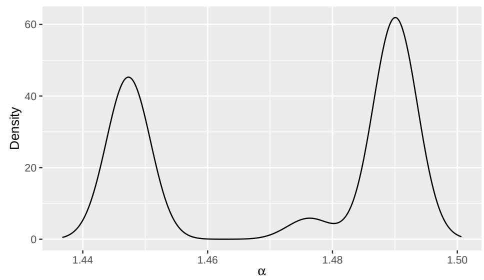

Fig. 9. Marginal Posterior function plot for α

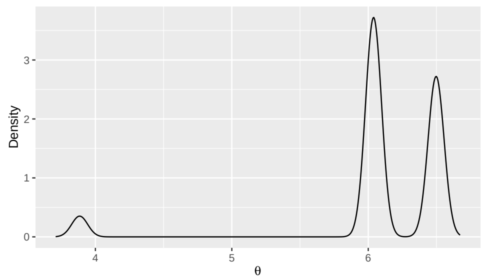

Fig. 10. Marginal Posterior function plot for θ

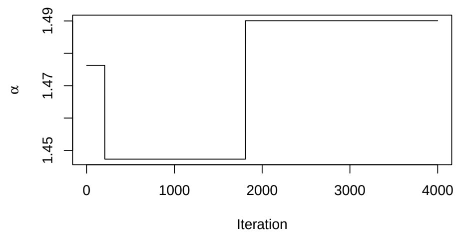

Fig. 11. Trace plot for α

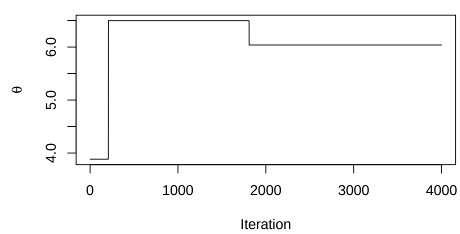

Fig. 12. Trace plot for θ

The above plots represent the Metropolis-Hastings Markov chain Monte Carlo (MCMC) algorithm to estimate the parameters of the SHCJ distribution based on a hypothetical dataset. The ACF plots in Figs. 5 and 6 display the autocorrelation of the sampled values at different lags. Autocorrelation measures the correlation between the current value of a sample and its previous values. In an MCMC chain, high autocorrelation indicates slow mixing and poor exploration of the parameter space. Low autocorrelation is desired, where the values quickly decay to zero or are within a reasonable range. In the ACF plots however, quick decay is not tenable. The CDF plots in figures 7 and 8 show the estimated cumulative distribution functions for the parameters θ and α. The estimated CDF closely matches the true distribution in figure 14. The shape of the Marginal posterior densities in figures 9 and 10 indicate the precision of the parameter estimation hence help to understand the central tendency as well as the variability of the posterior distribution. The trace plots show the evolution of the sampled values of the parameters θ and α over the iterations. From figures 11 and 12, the trace plots do not exhibit absolute convergence due to sudden fluctuation and notable trend. However, there is proper mixing which means that the chain explores the full range of possible parameter values.

The following inference can be drawn from Table 5 to Table 8.

1. As the sample size increases, all estimators' Bias and RMSE values fall, demonstrating improved accuracy in model parameter estimation.
2. The least biased parameters for classical estimation of the parameters at various sample sizes are WLSE, ADE, and RTADE. While for bayesian estimation, the Linear-Exponential Loss function produced the least bias.
3. For all sample sizes, the estimators' bias is positive.

8 Real Data Application

The following data represent 40 patients suffering from blood cancer (leukemia) from one of Ministry of Health Hospitals in Saudi Arabia studied by Abouammoh et al. [33]

Table 10. Data represent 40 patients suffering from blood cancer (leukemia) from one of Ministry of Health Hospitals in Saudi Arabia

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.37	2.532	2.693	2.805	2.91	2.912	3.192	3.263	3.348
3.348	3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244
4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	5.381

The usefulness of the new distribution is illustrated here using some analytical measures namely negative Log-Likelihood (-LL), Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and the Hannan–Quinn information criterion (HQIC) while Kolmogorov-Smirnov (K-S) statistic with its associated p-value is used to measure fitness of the distribution to the given data. Five competing distributions namely Chris-Jerry by Onyekwere and Obulezi [34], Lindley by Lindley [2], Pareto by Newman [37], Two-Parameter Odoma by Enogwe et al. [36] and Generalized Inverted Exponential by Krishna [35] are fitted to the data and compared with the proposed distribution, see Table 11.

From tabel 11, Figs. 13 to 15, the proposed distribution fits the data better and the model performance is also preferred to the other distributions considered. The PP plot for the proposed distribution shows better fit than the other competing distributions. Majority of the data points fall in the straight line hence the usefulness of Shifted Chris-Jerry distribution is obvious. Again, the proposed distribution perform better than the parent distribution name Chris-Jerry distribution, see the last plot in Fig. 17.

Table 11. MLE estimates and Analytical Measures for the fitted distributions based on blood cancer data

Dist.	Para	Estimate	-LL	AIC	CAIC	BIC	HQIC	K-S	P-value
SHCJ	θ	0.8739	73.82	149.6339	149.7391	151.3270	150.2445	0.1929	0.1019
	α	0.3150							
LD	θ	0.5276	80.25	162.5013	162.6065	164.1901	163.1119	0.2411	0.0195
PD	θ	97.3209	85.92	176.5905	176.9148	179.9682	177.8117	0.3035	0.0013
	α	31.6109							
TPOD	θ	1.0992	71.05	157.1813	157.5056	160.5590	158.4026	0.2429	0.0178
	α	3.6468							
GIE	θ	2.6504	85.84	175.6988	176.0231	179.0765	176.9201	0.2315	0.0266
	α	3.5787							
CJ	θ	0.8024	77.07	156.1463	156.2515	157.8352	156.7569	0.2147	0.0510

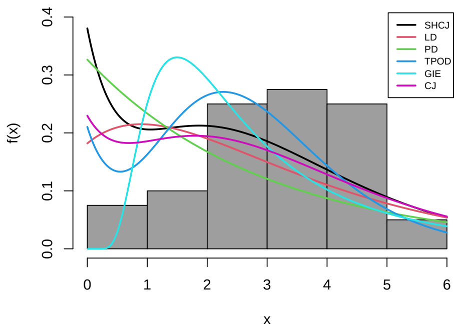

Fig. 13. Empirical and Theoretical PDF

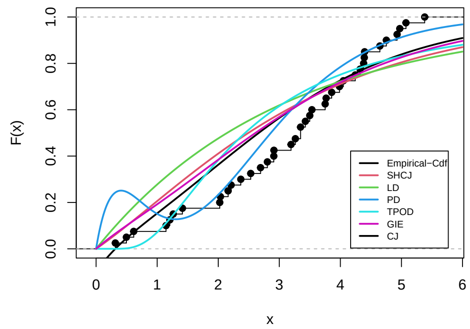

Fig. 14. Empirical and Theoretical CDF

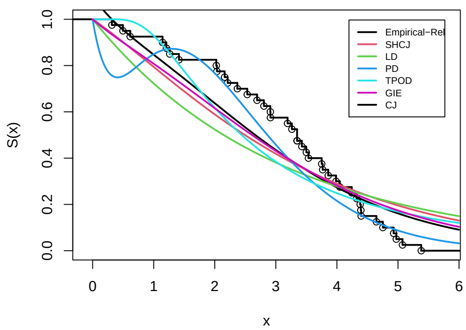

Fig. 15. Empirical and Theoretical Reliability

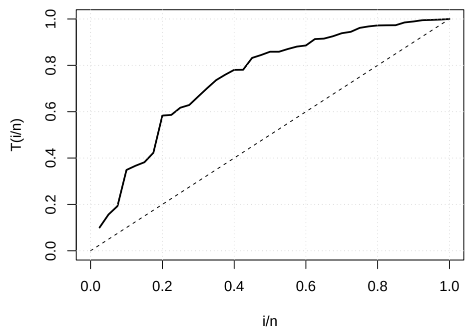

Fig. 16. TTT Plot

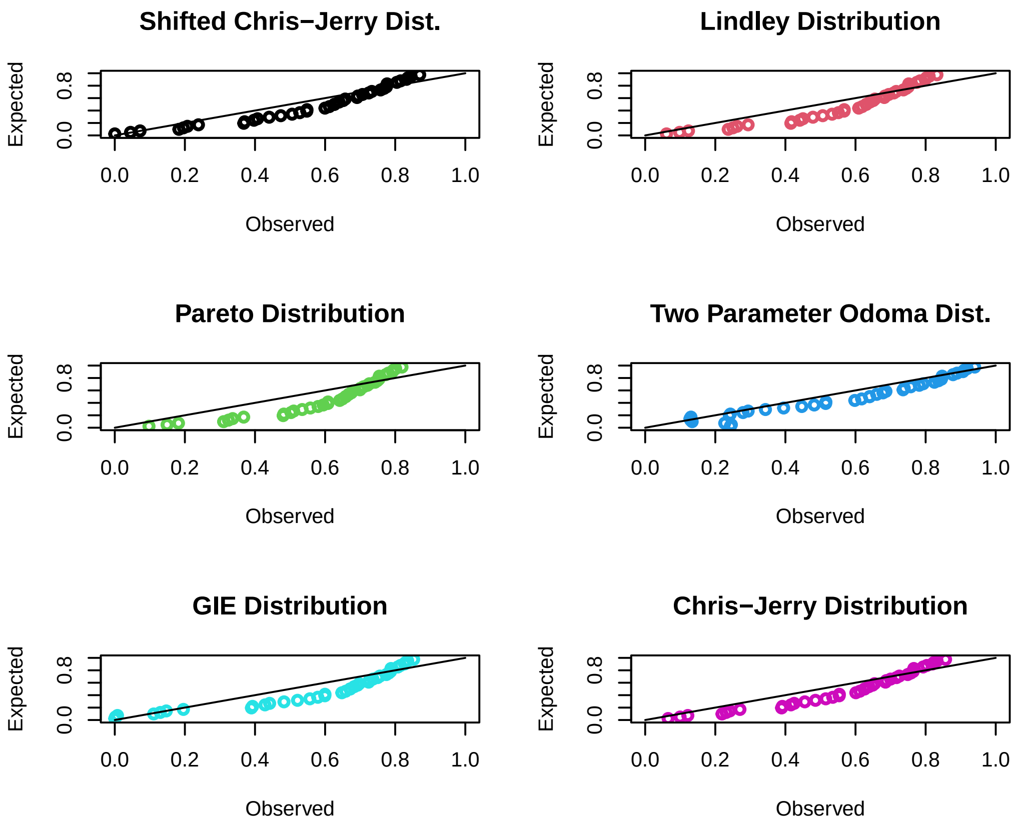

Fig. 17. PP plots

9 Conclusion

This simulation study considered the following classical methods namely maximum likelihood, maximum product spacing, least squares, weighted least squares, Cramer Von Mises, Anderson Darling and right-tailed Anderson Darling estimation and bayesian estimation under squared error loss, linear-exponential loss and generalized entropy loss functions. From the single acceptance sampling plans in Table 1 to Table 4, we deduce that as α^* and c increase, the sample size n increases leading to a decrease in the $L(p_0)$. As a increases, the required sample size n decreases and $L(p_0)$ increases. Again, as θ increases and γ is fixed, the required sample size n increases and $L(p_0)$ decreases. From the Tables 5 to 8, being the classical and bayesian simulation study, for various parameter values as the sample size increases, the Bias and Root Mean Squared error (RMSE) all fall demonstrating improved accuracy in model parameter estimation. The least biased parameters for classical estimation of the parameters at various sample sizes are WLSE, ADE, and RTADE. While for bayesian estimation, the Linear-Exponential Loss function produced the least bias. For all sample sizes, the estimators' bias is positive. In the ACF plots, quick decay is not tenable. The CDF plots in Figs. 7 and 8 show the estimated cumulative distribution functions for the parameters θ and α. The estimated CDF closely matches the true distribution in Fig. 14. The shape of the Marginal posterior densities in Figs. 9 and 10 indicate the precision of the parameter estimation hence help to understand the central tendency as well as the variability of the posterior distribution. From Figs. 11 and 12, the trace plots do not exhibit absolute convergence due to sudden fluctuation and notable trend. However, there is proper mixing which means that the chain explores the full range of possible parameter values. The application discussed shows that the proposed model fits the data better and also perform better in estimation of the parameters.

Competing Interests

Authors have declared that no competing interests exist.

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