Asian Journal of Probability and Statistics

15(4): 54-75, 2021; Article no.AJPAS.76935

ISSN: 2582-0230

The Second Hyper-Zagreb Index of Complement Graphs and Its Applications of Some Nano Structures

Mohammed Alsharafi[1*], Yusuf Zeren[1] and Abdu Alameri[2]

[1] *Department of Mathematics, Faculty of Arts and Science, Yildiz Technical University, Istanbul, Turkey.*

[2] *Department of Biomedical Engineering, Faculty of Engineering, University of Science and Technology, Yemen.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2021/v15i430364

Editor(s):

(1) Dr. Manuel Alberto M. Ferreira, Lisbon University, Portugal.

Reviewers:

(1) Mona Mohamed Mohamed Ahmed Khalil, Al-Azahar University, Egypt.

(2) Farhana Yasmeen, COMSATS University Islamabad, Pakistan.

Complete Peer review History, details of the editor(s), Reviewers and additional Reviewers are available here: https://www.sdiarticle5.com/review-history/76935

Received: ***20 September 2021***

Accepted: ***17 November 2021***

Published: ***30 November 2021***

Original Research Article

Abstract

In chemical graph theory, a topological descriptor is a numerical quantity that is based on the chemical structure of underlying chemical compound. Topological indices play an important role in chemical graph theory especially in the quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR). In this paper, we present explicit formulae for some basic mathematical operations for the second hyper-Zagreb index of complement graph containing the join $G_1 + G_2$, tensor product $G_1 \otimes G_2$, Cartesian product $G_1 \times G_2$, composition $G_1 \circ G_2$, strong product $G_1 * G_2$, disjunction $G_1 \vee G_2$ and symmetric difference $G_1 \oplus G_2$. Moreover, we studied the second hyper-Zagreb index for some certain important physicochemical structures such as molecular complement graphs of V-Phenylenic Nanotube $VPHX[q,p]$, V-Phenylenic Nanotorus $VPHY[m,n]$ and Titania Nanotubes TiO_2.

Keywords: *Zagreb index; Hyper-Zagreb index; Y-index; Second Hyper-Zagreb index; Forgotten index; graph operation; Titania Nanotubes; V-Phenylenic Nanotube $VPHX[q,p]$; V-Phenylenic Nanotorus $VPHY[m,n]$; complement graph.*

**Corresponding author: E-mail: alsharafi205010@gmail.com;*

2010 Mathematics Subject Classification: 05C76; 05C07; 92E10.

1 Introduction

A graph can be recognized by a numeric value which represent the whole graph. Topological indices are numerical quantity which are used to correlate the physico-chemical properties of a molecule with its structure such as boiling point, stability, strain energy etc. There are a wide range of topological descriptors which based on degree-based, distance-based, eigenvalue-based, matching-based and mixed-based. The topological indices which based on degree-based play a vital in chemical sciences and computing these topological indices is one of the recent areas of research in chemical graph theory. In our study, we have computed the degree-based topological indices of some graph operations and some nanostructure. All graphs in this paper are finite and simple, let G be a finite simple graph on $V(G) = n$, vertices and $E(G) = m$, edges, the degree of a vertex v is the number of edges incident to v, denoted by $\delta_G(v)$. The first and second Zagreb indices have been introduced by Gutman and Trinajstić in 1972 [1]. They are respectively defined as:

$$M_1(G) = \sum_{v\in V(G)} \delta_G^2(v) = \sum_{uv\in E(G)} [\delta_G(u) + \delta_G(v)],$$

$$M_2(G) = \sum_{uv\in E(G)} \delta_G(u)\,\delta_G(v).$$

The first and second Zagreb coindices have been introduced by A.R. Ashrafi, T. Došlić, and A. Hamzeh in 2010 [2]. They are respectively defined as:

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} [\delta_G(u) + \delta_G(v)], \qquad \overline{M}_2(G) = \sum_{uv\notin E(G)} \delta_G(u)\,\delta_G(v),$$

In 2013, Shirdel et al. [3] introduced degree-based of Zagreb indices named Hyper-Zagreb index which is defined as:

$$HM(G) = \sum_{uv\in E(G)} (\delta_G(u) + \delta_G(v))^2,$$

In 2016, M. Veylaki et al. [4] introduced degree-based of Zagreb coindex named Hyper-Zagreb coindex which is defined as :

$$\overline{HM}(G) = \sum_{uv\notin E(G)} (\delta_G(u) + \delta_G(v))^2,$$

Furtula and Gutman in (2015) introduced forgotten index (F-index) [5]. which defined as:

$$F(G) = \sum_{uv\in E(G)} \left({\delta_G}^2(u) + {\delta_G}^2(v)\right),$$

Furtula et al. in (2015) defined forgotten coindex (F-coindex)[6] as the following:

$$\overline{F}(G) = \sum_{uv\notin E(G)} \left({\delta_G}^2(u) + {\delta_G}^2(v)\right),$$

Ranjini et al. [7] defined the redefined first, second and third Zagreb indices for a graph G and these are manifested as:

$$ReZG_1(G) = \sum_{uv\in E(G)} \frac{\delta(u)+\delta(v)}{\delta(u)\delta(v)}, \quad ReZG_2(G) = \sum_{uv\in E(G)} \frac{\delta(u)\delta(v)}{\delta(u)+\delta(v)},$$

$$ReZG_3(G) = \sum_{uv \in E(G)} (\delta_G(u)\, \delta_G(v))(\delta_G(u) + \delta_G(v)),$$

In 2020, Alameri et al. [8],[9] presented new degree-based of Zagreb index and coindex denoted by (Y-index, Y-coindex) and defined respectively as:

$$Y(G) = \sum_{u \in V(G)} \delta_G^4(u) = \sum_{uv \in E(G)} [\delta_G^3(u) + \delta_G^3(v)],$$

$$\overline{Y}(G) = \sum_{uv \notin E(G)} [\delta_G^3(u) + \delta_G^3(v)],$$

In 2016, Wei Gao et al [10] defined the second Hyper-Zagreb index of a graph G, which is defined as:

$$HM_2(G) = \sum_{uv \in E(G)} (\delta_G(u)\, \delta_G(v))^2,$$

Therefore, Wei Gao et al. [10] in (2016) computed the second Hyper-Zagreb index $HM_2(G)$ of Carbon Nanocones $CNC_k[n]$. M. Farahani et al. [11] studied the second Hyper-Zagreb index of Molecular graphs V-Phenylenic Nanotubes and V-Phenylenic Nanotorus. A. Modabish et al. [12] in (2021) computed the second Hyper-Zagreb index of some special graph and graph operation. Here we continue this line of research by exploring the behavior of the second Hyper-Zagreb index under several important operations such as the tensor product $\overline{G_1 \otimes G_2}$, join $G_1 + G_2$ and $\overline{G_1 + G_2}$, strong product $\overline{G_1 * G_2}$, Cartesian product $\overline{G_1 \times G_2}$, composition $\overline{G_1 \circ G_2}$, disjunction $\overline{G_1 \vee G_2}$ and symmetric difference $\overline{G_1 \oplus G_2}$. The results are applied on some molecular complement graphs such as of V-Phenylenic Nanotube $VPHX[q,p]$, V-Phenylenic Nanotorus $VPHY[m,n]$ and Titania Nanotubes TiO_2. Any unexplained terminology is standard, typically as in [10],[11],[13],[14],[15],[16],[17], and [18].

2 Preliminaries

Definition 2.1. [25] The complement graph of a graph G denoted by $\overline{G}$, is a graph on the same set of vertices $V(G)$ in which two vertices u and v are adjacent, i.e., connected by an edge uv, if and only if they are not adjacent in G. Hence, $uv \in E(\overline{G}), \iff uv \notin E(G)$. Obviously $E(G) \cup E(\overline{G}) = E(K_n)$, and $\overline{m} = |E(\overline{G})| = \binom{n}{2} - m$, the degree of a vertex u in $\overline{G}$, is the number of edges incident to u, denoted by $\delta_{\overline{G}}(u) = (n-1) - \delta_G(u)$.

Theorem 2.1. *[11],[19] The first and second Zagreb and Hyper-Zagreb indices of the V-Phenylenic Nanotubes $VPHX[q,p] (\forall q, p \in \mathbb{N} - \{1\})$ (Fig. 1) are given by,*

$$M_1(VPHX[q,p]) = 54pq - 10q, \qquad M_2(VPHX[q,p]) = 81pq + 3q,$$

$$HM(VPHX[q,p]) = 4q(81p - 20), \qquad HM_2(VPHX[q,p]) = 9q(81p - 29).$$

Theorem 2.2. *[11],[19] The Zagreb and Hyper-Zagreb indices of the V-Phenylenic Nanotorus $VPHY[m,n]$ $(\forall m, n \in \mathbb{N} - \{1\})$ (Fig. 3) are given by,*

$$M_1(VPHY[m,n]) = 54mn, \qquad M_2(VPHY[m,n]) = 81mn,$$

$$HM(VPHY[m,n]) = 324mn, \qquad HM_2(VPHY[m,n]) = 729mn.$$

3 Discussion and Main Results

In this section, we study the second Hyper-Zagreb index of various complement graph operations and compute HM_2 for some certain important Nano-Structures such as molecular complement graphs of V-Phenylenic Nanotube $VPHX[q,p]$, V-Phenylenic Nanotorus $VPHY[m,n]$ and Titania Nanotubes TiO_2.

3.1 Second Hyper-Zagreb index of complement graph binary operations

Theorem 3.1. *Let G be a graph with n vertices and m edges. Then,*

$$\begin{aligned} HM_2(\overline{G}) &= m(n-1)^4 - 2(n-1)^3 M_1(G) + (n-1)^2 F(G) + 4(n-1)^2 M_2(G) \\ &- 2(n-1)M_1(G)F(G) + HM_2(G). \end{aligned}$$

Proof. By definition of the second Hyper-Zagreb index, we have

$$\begin{aligned} HM_2(\overline{G}) &= \sum_{uv\notin E(G)} [\delta_{\overline{G}}(u)\delta_{\overline{G}}(v)]^2 \\ &= \sum_{uv\in E(G)} [(n-1)-\delta_G(u)]^2[(n-1)-\delta_G(v)]^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{uv\in E(G)} \big[[(n-1)^2 - 2(n-1)\delta_G(u) + \delta_G^2(u)][(n-1)^2 - 2(n-1)\delta_G(v) + \delta_G^2(v)]\big] \\
&= \sum_{uv\in E(G)} \big[(n-1)^2[(n-1)^2 - 2(n-1)\delta_G(v) + \delta_G^2(v)] \\
&\quad -2(n-1)\delta_G(u)[(n-1)^2 - 2(n-1)\delta_G(v) + \delta_G^2(v)] \\
&\quad +\delta_G^2(u)[(n-1)^2 - 2(n-1)\delta_G(v) + \delta_G^2(v)]\big] \\
&= \sum_{uv\in E(G)} \big[(n-1)^4 - 2(n-1)^3\delta_G(v) + (n-1)^2\delta_G^2(v) - 2(n-1)^3\delta_G(u) \\
&+ 4(n-1)^2\delta_G(u)\delta_G(v) - 2(n-1)\delta_G(u)\delta_G^2(v) + (n-1)^2\delta_G^2(u) \\
&\quad -2(n-1)\delta_G^2(u)\delta_G(v) + \delta_G^2(u)\delta_G^2(v)\big] \\
&= \sum_{uv\in E(G)} \big[(n-1)^4 - 2(n-1)^3\delta_G(v) - 2(n-1)^3\delta_G(u) + (n-1)^2\delta_G^2(v) \\
&\quad +(n-1)^2\delta_G^2(u) + 4(n-1)^2\delta_G(u)\delta_G(v) - 2(n-1)\delta_G(u)\delta_G^2(v) \\
&\quad -2(n-1)\delta_G^2(u)\delta_G(v) + \delta_G^2(u)\delta_G^2(v)\big] \\
&= (n-1)^4 \sum_{uv\in E(G)} 1 - 2(n-1)^3 \sum_{uv\in E(G)} [\delta_G(v) + \delta_G(u)] \\
&+ (n-1)^2 \sum_{uv\in E(G)} [\delta_G^2(v) + \delta_G^2(u)] + 4(n-1)^2 \sum_{uv\in E(G)} \delta_G(u)\delta_G(v) \\
&- 2(n-1) \sum_{uv\in E(G)} [\delta_G(u)\delta_G^2(v) + \delta_G^2(u)\delta_G(v)] + \sum_{uv\in E(G)} \delta_G^2(u)\delta_G^2(v) \\
&= m(n-1)^4 - 2(n-1)^3 M_1(G) + (n-1)^2 F(G) + 4(n-1)^2 M_2(G) \\
&- 2(n-1)M_1(G)F(G) + HM_2(G).
\end{aligned}$$

□

Definition 3.1. [20],[21] The tensor product $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1)\times V(G_2)$ and $E(G_1\otimes G_2) = (u_1,u_2)(v_1,v_2)|u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$, such that $|V(G_1 \otimes G_2)| = n_1n_2$, $|E(G_1 \otimes G_2)| = 2m_1m_2$ and $\delta_{G_1\otimes G_2}(u,v) = \delta_{G_1}(u)\delta_{G_2}(v)$.

Theorem 3.2. *The second Hyper-Zagreb index of complement $G_1 \otimes G_2$ is given by:*

$$\begin{aligned} HM_2(\overline{G_1 \otimes G_2}) &= 2m_1m_2(n_1n_2-1)^4 - 2(n_1n_2-1)^3M_1(G_1)M_1(G_2) \\ &+ (n_1n_2-1)^2F(G_1)F(G_2) + 8(n_1n_2-1)^2M_2(G_1)M_2(G_2) \\ &- 2(n_1n_2-1)M_1(G_1)M_1(G_2)F(G_1)F(G_2) + 2HM_2(G_1)HM_2(G_2). \end{aligned}$$

Proof. By (Theorem 3.1) we have

$$\begin{aligned} HM_2(\overline{G}) &= m(n-1)^4 - 2(n-1)^3M_1(G) + (n-1)^2F(G) + 4(n-1)^2M_2(G) \\ &- 2(n-1)M_1(G)F(G) + HM_2(G). \end{aligned}$$

And $M_1(G_1 \otimes G_2) = M_1(G_1)M_1(G_2)$, $M_2(G_1 \otimes G_2) = 2M_2(G_1)M_2(G_2)$ given in [22]. $F(G_1 \otimes G_2) = F(G_1)F(G_2)$ given in [23]. $HM_2(G_1 \otimes G_2) = 2HM_2(G_1)HM_2(G_2)$ given in [12]. and since $|E(G_1 \otimes G_2)| = 2m_1m_2$, $|V(G_1 \otimes G_2)| = n_1n_2$. Then,

$$\begin{aligned} HM_2(\overline{G_1 \otimes G_2}) &= |E(G_1 \otimes G_2)|(|V(G_1 \otimes G_2)|-1)^4 - 2(|V(G_1 \otimes G_2)|-1)^3M_1(G_1 \otimes G_2) \\ &+ (|V(G_1 \otimes G_2)|-1)^2F(G_1 \otimes G_2) + 4(|V(G_1 \otimes G_2)|-1)^2M_2(G_1 \otimes G_2) \\ &- 2(|V(G_1 \otimes G_2)|-1)M_1(G_1 \otimes G_2)F(G_1 \otimes G_2) + HM_2(G_1 \otimes G_2) \\ &= 2m_1m_2(n_1n_2-1)^4 - 2(n_1n_2-1)^3M_1(G_1)M_1(G_2) \\ &+ (n_1n_2-1)^2F(G_1)F(G_2) + 8(n_1n_2-1)^2M_2(G_1)M_2(G_2) \\ &- 2(n_1n_2-1)M_1(G_1)M_1(G_2)F(G_1)F(G_2) + 2HM_2(G_1)HM_2(G_2). \end{aligned}$$

□

Definition 3.2. [20],[21] The join $G_1 + G_2$ of two graphs G_1 and G_2 is a graph with vertex set $V(G_1+G_2) = V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)\cup\{uv|u \in V(G_1), v \in V(G_2)\}$, such that $|V(G_1+G_2)| = n_1+n_2$, $|E(G_1+G_2)| = m_1+m_2+n_1n_2$ and $\delta_{G_1+G_2}(a) = \begin{cases} \delta_{G_1}(a) + n_2 & a \in V(G_1) \\ \delta_{G_2}(a) + n_1 & a \in V(G_2) \end{cases}$.

Theorem 3.3. *[25] The second Zagreb index of $G_1 + G_2$ is given by:*

$$\begin{aligned} M_2(G_1+G_2) &= M_2(G_1) + M_2(G_2) + n_2M_1(G_1) + n_1M_1(G_2) + n_2^2m_1 + n_1^2m_2 \\ &+ 4m_1m_2 + 2m_1n_1n_2 + 2m_2n_2n_1 + n_1^2n_2^2. \end{aligned}$$

Theorem 3.4. *The second Hyper-Zagreb index of $G_1 + G_2$ is given by:*

$$\begin{aligned} HM_2(G_1+G_2) &= n_2^2F(G_1) + n_1^2F(G_2) + 4n_2^2M_2(G_1) + 4n_1^2M_2(G_2) \\ &+2n_2^3M_1(G_1) + 2n_1^3M_1(G_2) + n_2^4m_1 + n_1^4m_2 + n_1^3n_2^3 \\ &+[4n_1m_2 + n_1^2n_2]M_1(G_1) + [4n_2m_1 + n_1n_2^2]M_1(G_2) \\ &+4n_1^2n_2^2[m_1+m_2] + 16n_1n_2m_1m_2 + M_1(G_1)M_1(G_2) \\ &+2n_2M_1(G_1)F(G_1) + 2n_1M_1(G_2)F(G_2) + HM_2(G_1) + HM_2(G_2). \end{aligned}$$

Proof. By definitions of the second Hyper-Zagreb index, we have

$$\begin{aligned} HM_2(G_1+G_2) &= \sum_{uv\in E(G_1+G_2)} [\delta^2_{G_1+G_2}(u)\delta^2_{G_1+G_2}(v)] \\ &= \sum_{uv\in E(G_1)} [\delta^2_{G_1+G_2}(u)\delta^2_{G_1+G_2}(v)] + \sum_{uv\in E(G_2)} [\delta^2_{G_1+G_2}(u)\delta^2_{G_1+G_2}(v)] \\ &+ \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} [\delta^2_{G_1+G_2}(u)\delta^2_{G_1+G_2}(v)] \\ &= \sum_{uv\in E(G_1)} [(\delta_{G_1}(u)+n_2)^2(\delta_{G_1}(v)+n_2)^2] \\ &+ \sum_{uv\in E(G_2)} [(\delta_{G_2}(u)+n_1)^2(\delta_{G_2}(v)+n_1)^2] \\ &+ \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} [(\delta_{G_1}(u)+n_2)^2(\delta_{G_2}(v)+n_1)^2] \end{aligned}$$

$$\begin{aligned} &= \sum_{uv\in E(G_1)} [(\delta^2_{G_1}(u)+2n_2\delta_{G_1}(u)+n_2^2)(\delta^2_{G_1}(v)+2n_2\delta_{G_1}(v)+n_2^2)] \\ &+ \sum_{uv\in E(G_2)} [(\delta^2_{G_2}(u)+2n_1\delta_{G_2}(u)+n_1^2)(\delta^2_{G_2}(v)+2n_1\delta_{G_2}(v)+n_1^2)] \\ &+ \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} [(\delta^2_{G_1}(u)+2n_2\delta_{G_1}(u)+n_2^2)(\delta^2_{G_2}(v)+2n_1\delta_{G_2}(v)+n_1^2)] \\ &= \sum_{uv\in E(G_1)} \big[\delta^2_{G_1}(u)\delta^2_{G_1}(v)+2n_2\delta^2_{G_1}(u)\delta_{G_1}(v)+n_2^2\delta^2_{G_1}(u)+2n_2\delta_{G_1}(u)\delta^2_{G_1}(v) \\ &+4n_2^2\delta_{G_1}(u)\delta_{G_1}(v)+2n_2^3\delta_{G_1}(u)+n_2^2\delta^2_{G_1}(v)+2n_2^3\delta_{G_1}(v)+n_2^4\big] \\ &+ \sum_{uv\in E(G_2)} \big[\delta^2_{G_2}(u)\delta^2_{G_2}(v)+2n_1\delta^2_{G_2}(u)\delta_{G_2}(v)+n_1^2\delta^2_{G_2}(u)+2n_1\delta_{G_2}(u)\delta^2_{G_2}(v) \\ &+4n_1^2\delta_{G_2}(u)\delta_{G_2}(v)+2n_1^3\delta_{G_2}(u)+n_1^2\delta^2_{G_2}(v)+2n_1^3\delta_{G_2}(v)+n_1^4\big] \\ &+ \sum_{u\in V(G_1)}\sum_{v\in V(G_2)} \big[\delta^2_{G_1}(u)\delta^2_{G_2}(v)+2n_1\delta^2_{G_1}(u)\delta_{G_2}(v)+n_1^2\delta^2_{G_1}(u) \\ &+2n_2\delta_{G_1}(u)\delta^2_{G_2}(v)+4n_1n_2\delta_{G_1}(u)\delta_{G_2}(v)+2n_1^2n_2\delta_{G_1}(u)+n_2^2\delta^2_{G_2}(v) \\ &+2n_1n_2^2\delta_{G_2}(v)+n_1^2n_2^2\big] \end{aligned}$$

Step 1

$$\begin{aligned} \sum_{uv\in E(G_1)} [...] &= \sum_{uv\in E(G_1)} [\delta^2_{G_1}(u)\delta^2_{G_1}(v)] + 2n_2 \sum_{uv\in E(G_1)} [\delta^2_{G_1}(u)\delta_{G_1}(v)+\delta_{G_1}(u)\delta^2_{G_1}(v)] \\ &+n_2^2 \sum_{uv\in E(G_1)} [\delta^2_{G_1}(u)+\delta^2_{G_1}(v)] + 4n_2^2 \sum_{uv\in E(G_1)} [\delta_{G_1}(u)\delta_{G_1}(v)] \\ &+2n_2^3 \sum_{uv\in E(G_1)} [\delta_{G_1}(u)+\delta_{G_1}(v)] + n_2^4 \sum_{uv\in E(G_1)} (1) \\ &= HM_2(G_1)+2n_2M_1(G_1)F(G_1)+n_2^2F(G_1)+4n_2^2M_2(G_1) \\ &+2n_2^3M_1(G_1)+n_2^4m_1. \end{aligned}$$

Step 2

$$\begin{aligned}\sum_{uv\in E(G_2)}[...] &= \sum_{uv\in E(G_2)}[\delta_{G_2}^2(u)\delta_{G_2}^2(v)] + 2n_1\sum_{uv\in E(G_2)}[\delta_{G_2}^2(u)\delta_{G_2}(v)+\delta_{G_2}(u)\delta_{G_2}^2(v)]\\ &+n_1^2\sum_{uv\in E(G_2)}[\delta_{G_2}^2(u)+\delta_{G_2}^2(v)] + 4n_1^2\sum_{uv\in E(G_2)}[\delta_{G_2}(u)\delta_{G_2}(v)]\\ &+2n_1^3\sum_{uv\in E(G_2)}[\delta_{G_2}(u)+\delta_{G_2}(v)] + n_1^4\sum_{uv\in E(G_2)}(1)\\ &= HM_2(G_2)+2n_1M_1(G_2)F(G_2)+n_1^2F(G_2)+4n_1^2M_2(G_2)\\ &+2n_1^3M_1(G_2)+n_1^4m_2.\end{aligned}$$

Step 3

$$\begin{aligned}\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}[...] &= \sum_{u\in V(G_1)}\sum_{v\in V(G_2)}[\delta_{G_1}^2(u)\delta_{G_2}^2(v)] + 2n_1\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}[\delta_{G_1}^2(u)\delta_{G_2}(v)]\\ &+n_1^2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\delta_{G_1}^2(u) + 2n_2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}[\delta_{G_1}(u)\delta_{G_2}^2(v)]\\ &+4n_1n_2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}[\delta_{G_1}(u)\delta_{G_2}(v)] + 2n_1^2n_2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\delta_{G_1}(u)\\ &+n_2^2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\delta_{G_2}^2(v) + 2n_1n_2^2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\delta_{G_2}(v)\\ &+n_1^2n_2^2\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}(1)\\ &= M_1(G_1)M_1(G_2)+4n_1m_2M_1(G_1)+n_1^2n_2M_1(G_1)+4n_2m_1M_1(G_2)\\ &+16n_1n_2m_1m_2+4m_1n_1^2n_2^2+n_1n_2^2M_1(G_2)+4m_2n_1^2n_2^2+n_1^3n_2^3.\end{aligned}$$

It is easy to see that the summation of step 1, step 2 and step 3 complete the proof. □

Theorem 3.5. *The second Hyper-Zagreb index of complement $G_1 + G_2$ is given by:*

$$\begin{aligned}HM_2(\overline{G_1+G_2}) &= (m_1+m_2+n_1n_2)(n_1+n_2-1)^4 - 2(n_1+n_2-1)^3[M_1(G_1)+M_1(G_2)\\ &+ n_1n_2^2+n_2n_1^2+4m_1n_2+4m_2n_1] + (n_1+n_2-1)^2[F(G_1)+F(G_2)\\ &+ 3n_2M_1(G_1)+3n_1M_1(G_2)+6n_2^2m_1+6n_1^2m_2+n_1n_2^3+n_2n_1^3]\\ &+ 4(n_1+n_2-1)^2[M_2(G_1)+M_2(G_2)+n_2M_1(G_1)+n_1M_1(G_2)\\ &+ n_2^2m_1+n_1^2m_2+4m_1m_2+2m_1n_1n_2+2m_2n_2n_1+n_1^2n_2^2]\\ &- 2(n_1+n_2-1)(M_1(G_1)+M_1(G_2)+n_1n_2^2+n_2n_1^2+4m_1n_2\\ &+ 4m_2n_1)\big[F(G_1)+F(G_2)+3n_2M_1(G_1)+3n_1M_1(G_2)\\ &+ 6n_2^2m_1+6n_1^2m_2+n_1n_2^3+n_2n_1^3]\big] + 2n_2M_1(G_1)F(G_1)\\ &+ 2n_1M_1(G_2)F(G_2)+HM_2(G_1)+HM_2(G_2)+n_2^2F(G_1)+n_1^2F(G_2)\\ &+ 4n_2^2M_2(G_1)+4n_1^2M_2(G_2)+2n_2^3M_1(G_1)+2n_1^3M_1(G_2)+n_2^4m_1\\ &+ n_1^4m_2+n_1^3n_2^3+[4n_1m_2+n_1^2n_2]M_1(G_1)+[4n_2m_1+n_1n_2^2]M_1(G_2)\\ &+ 4n_1^2n_2^2[m_1+m_2]+16n_1n_2m_1m_2+M_1(G_1)M_1(G_2).\end{aligned}$$

Proof. By (Theorem 3.1) we have

$$\begin{aligned}HM_2(\overline{G}) &= m(n-1)^4-2(n-1)^3M_1(G)+(n-1)^2F(G)+4(n-1)^2M_2(G)\\ &- 2(n-1)M_1(G)F(G)+HM_2(G).\end{aligned}$$

And $M_1(G_1+G_2) = M_1(G_1) + M_1(G_2) + n_1n_2^2 + n_2n_1^2 + 4m_1n_2 + 4m_2n_1$ given in [25]. $F(G_1 + G_2) = F(G_1) + F(G_2) + 3n_2M_1(G_1) + 3n_1M_1(G_2) + 6n_2^2m_1 + 6n_1^2m_2 + n_1n_2^3 + n_2n_1^3$ given in [23]. $|E(G_1+G_2)| = m_1 + m_2 + n_1n_2$, $\quad |V(G_1+G_2)| = n_1 + n_2$ and by (Theorem 3.3) and (Theorem 3.4). Then,

$$\begin{aligned} HM_2(\overline{G_1+G_2}) &= |E(G_1+G_2)|(|V(G_1+G_2)|-1)^4 - 2(|V(G_1+G_2)|-1)^3M_1(G_1+G_2) \\ &+ (|V(G_1+G_2)|-1)^2F(G_1+G_2) + 4(|V(G_1+G_2)|-1)^2M_2(G_1+G_2) \\ &- 2(|V(G_1+G_2)|-1)M_1(G_1+G_2)F(G_1+G_2) + HM_2(G_1+G_2) \end{aligned}$$

$$\begin{aligned} &= (m_1+m_2+n_1n_2)(n_1+n_2-1)^4 - 2(n_1+n_2-1)^3[M_1(G_1)+M_1(G_2) \\ &+ n_1n_2^2 + n_2n_1^2 + 4m_1n_2 + 4m_2n_1] + (n_1+n_2-1)^2[F(G_1)+F(G_2) \\ &+ 3n_2M_1(G_1) + 3n_1M_1(G_2) + 6n_2^2m_1 + 6n_1^2m_2 + n_1n_2^3 + n_2n_1^3] \\ &+ 4(n_1+n_2-1)^2[M_2(G_1) + M_2(G_2) + n_2M_1(G_1) + n_1M_1(G_2) \\ &+ n_2^2m_1 + n_1^2m_2 + 4m_1m_2 + 2m_1n_1n_2 + 2m_2n_2n_1 + n_1^2n_2^2] \\ &- 2(n_1+n_2-1)(M_1(G_1) + M_1(G_2) + n_1n_2^2 + n_2n_1^2 + 4m_1n_2 \\ &+ 4m_2n_1)\big[F(G_1) + F(G_2) + 3n_2M_1(G_1) + 3n_1M_1(G_2) \\ &+ 6n_2^2m_1 + 6n_1^2m_2 + n_1n_2^3 + n_2n_1^3]\big] + 2n_2M_1(G_1)F(G_1) \\ &+ 2n_1M_1(G_2)F(G_2) + HM_2(G_1) + HM_2(G_2) + n_2^2F(G_1) + n_1^2F(G_2) \\ &+ 4n_2^2M_2(G_1) + 4n_1^2M_2(G_2) + 2n_2^3M_1(G_1) + 2n_1^3M_1(G_2) + n_2^4m_1 \\ &+ n_1^4m_2 + n_1^3n_2^3 + [4n_1m_2 + n_1^2n_2]M_1(G_1) + [4n_2m_1 + n_1n_2^2]M_1(G_2) \\ &+ 4n_1^2n_2^2[m_1+m_2] + 16n_1n_2m_1m_2 + M_1(G_1)M_1(G_2). \end{aligned}$$

□

Definition 3.3. [20],[21]The strong product $G_1 * G_2$ of two graphs G_1 and G_2 is a graph with vertex set $V(G_1 * G_2) = V(G_1) \times V(G_2)$ and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if { $u_1 = u_2 \in V(G_1)$ and $v_1v_2 \in E(G_2)$ } or { $v_1 = v_2 \in V(G_2)$ and $u_1u_2 \in E(G_1)$}, such that $|V(G_1 * G_2)| = n_1n_2$, $|E(G_1 * G_2)| = m_1n_2 + n_1m_2 + 2m_1m_2$ and $\delta_{G_1*G_2}(u,v) = \delta_{G_1}(u) + \delta_{G_2}(v) + \delta_{G_1}(u)\delta_{G_2}(v)$.

Theorem 3.6. *The second Hyper-Zagreb index of complement $G_1 * G_2$ is given by:*

$$\begin{aligned}
HM_2(\overline{G_1 * G_2}) &= [m_1n_2 + n_1m_2 + 2m_1m_2](n_1n_2 - 1)^4 \\
&- 2(n_1n_2 - 1)^3[(n_2 + 6m_2)M_1(G_1) + 8m_2m_1 + (6m_1 + n_1)M_1(G_2) \\
&+ 2M_1(G_1)M_1(G_2)] + (n_1n_2 - 1)^2[n_2F(G_1) + n_1F(G_2) \\
&+ F(G_1)F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2) + 6m_2F(G_1) \\
&+ 6m_1F(G_2) + 3F(G_2)M_1(G_1) + 3F(G_1)M_1(G_2) + 6M_1(G_1)M_1(G_2)] \\
&+ 4(n_1n_2 - 1)^2[(n_2 + 5m_2)M_2(G_1) + (n_1 + 5m_1)M_2(G_2) + 3m_2M_1(G_1) \\
&+ 3m_1M_1(G_2) + 3M_1(G_1)M_1(G_2) + 2M_1(G_2)M_2(G_1) + 2M_1(G_1)M_2(G_2) \\
&+ M_2(G_1)M_2(G_2)] - 2(n_1n_2 - 1)[(n_2 + 6m_2)M_1(G_1) + 8m_2m_1 \\
&+ (6m_1 + n_1)M_1(G_2) + 2M_1(G_1)M_1(G_2)][n_2F(G_1) + n_1F(G_2) \\
&+ F(G_1)F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2) + 6m_2F(G_1) + 6m_1F(G_2) \\
&+ 3F(G_2)M_1(G_1) + 3F(G_1)M_1(G_2) + 6M_1(G_1)M_1(G_2)] \\
&+ HM_2(G_2)[n_1 + 10m_1 + 10M_1(G_1) + 8M_2(G_1) \\
&+ 6F(G_1) + 4ReZG_3(G_1) + Y(G_1)] + HM_2(G_1)[n_2 + 10m_2 \\
&+ 10M_1(G_2) + 8M_2(G_2) + 6F(G_2) + 4ReZG_3(G_2) + Y(G_2)] \\
&+ Y(G_2)[m_1 + 2M_1(G_1) + 4M_2(G_1) + F(G_1) + 2ReZG_3(G_1)] \\
&+ Y(G_1)[m_2 + 2M_1(G_2) + 4M_2(G_2) + F(G_2) + 2ReZG_3(G_2)] \\
&+ 4ReZG_3(G_2)[m_1 + 2M_1(G_1) + 2M_2(G_1) + 2F(G_1)] \\
&+ 4ReZG_3(G_1)[m_2 + 2M_1(G_2) + 2M_2(G_2) + 2F(G_2)] \\
&+ F(G_2)[3M_1(G_1) + 8M_2(G_1)] + F(G_1)[3M_1(G_2) + 8M_2(G_2)] \\
&+ 8M_2(G_1)M_2(G_2) + 4M_1(G_1)M_2(G_2) + 4M_1(G_2)M_2(G_1) \\
&+ 2HM_2(G_1)HM_2(G_2) + 5F(G_1)F(G_2) + 6ReZG_3(G_1)ReZG_3(G_2).
\end{aligned}$$

Proof. By using (Theorem 3.1) we have,

$$\begin{aligned}
HM_2(\overline{G_1 * G_2}) &= |E(G_1 * G_2)|(|V(G_1 * G_2)| - 1)^4 - 2(|V(G_1 * G_2)| - 1)^3M_1(G_1 * G_2) \\
&+ (|V(G_1 * G_2)| - 1)^2F(G_1 * G_2) + 4(|V(G_1 * G_2)| - 1)^2M_2(G_1 * G_2) \\
&- 2(|V(G_1 * G_2)| - 1)M_1(G_1 * G_2)F(G_1 * G_2) + HM_2(G_1 * G_2),
\end{aligned}$$

And by [15],[23],[12], respectively, we have

$$M_1(G_1 * G_2) = (n_2 + 6m_2)M_1(G_1) + 8m_2m_1 + (6m_1 + n_1)M_1(G_2) + 2M_1(G_1)M_1(G_2)$$

$$\begin{aligned}
M_2(G_1 * G_2) &= (n_2 + 5m_2)M_2(G_1) + (n_1 + 5m_1)M_2(G_2) + 3m_2M_1(G_1) + 3m_1M_1(G_2) \\
&+ 3M_1(G_1)M_1(G_2) + 2M_1(G_2)M_2(G_1) + 2M_1(G_1)M_2(G_2) + M_2(G_1)M_2(G_2).
\end{aligned}$$

$$\begin{aligned}
F(G_1 * G_2) &= n_2F(G_1) + n_1F(G_2) + F(G_1)F(G_2) + 6m_2M_1(G_1) \\
&+ 6m_1M_1(G_2) + 6m_2F(G_1) + 6m_1F(G_2) + 3F(G_2)M_1(G_1) \\
&+ 3F(G_1)M_1(G_2) + 6M_1(G_1)M_1(G_2).
\end{aligned}$$

$$\begin{aligned} HM_2(G_1 * G_2) &= HM_2(G_2)[n_1 + 10m_1 + 10M_1(G_1) + 8M_2(G_1) \\ &+ 6F(G_1) + 4ReZG_3(G_1) + Y(G_1)] + HM_2(G_1)[n_2 + 10m_2 \\ &+ 10M_1(G_2) + 8M_2(G_2) + 6F(G_2) + 4ReZG_3(G_2) + Y(G_2)] \\ &+ Y(G_2)[m_1 + 2M_1(G_1) + 4M_2(G_1) + F(G_1) + 2ReZG_3(G_1)] \\ &+ Y(G_1)[m_2 + 2M_1(G_2) + 4M_2(G_2) + F(G_2) + 2ReZG_3(G_2)] \\ &+ 4ReZG_3(G_2)[m_1 + 2M_1(G_1) + 2M_2(G_1) + 2F(G_1)] \\ &+ 4ReZG_3(G_1)[m_2 + 2M_1(G_2) + 2M_2(G_2) + 2F(G_2)] \\ &+ F(G_2)[3M_1(G_1) + 8M_2(G_1)] + F(G_1)[3M_1(G_2) + 8M_2(G_2)] \\ &+ 8M_2(G_1)M_2(G_2) + 4M_1(G_1)M_2(G_2) + 4M_1(G_2)M_2(G_1) \\ &+ 2HM_2(G_1)HM_2(G_2) + 5F(G_1)F(G_2) + 6ReZG_3(G_1)ReZG_3(G_2). \end{aligned}$$

And since $|E(G_1 * G_2)| = m_1n_2 + n_1m_2 + 2m_1m_2, \quad |V(G_1 * G_2)| = n_1n_2$. Which are complete the proof. □

Definition 3.4. [20],[21] The Cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 has the vertex set$V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(a, x)(b, y)$ is an edge of $G_1 \times G_2$ if $a = b$ and $xy \in E(G_2)$, or $ab \in E(G_1)$ and $x = y$, such that $|V(G_1 \times G_2)| = n_1n_2$, $|E(G_1 \times G_2)| = m_1n_2 + n_1m_2$, and $\delta_{G_1 \times G_2}(u, v) = \delta_{G_1}(u) + \delta_{G_2}(v)$.

Theorem 3.7. *The second Hyper-Zagreb index of complement $G_1 \times G_2$ is given by:*

$$\begin{aligned} HM_2(\overline{G_1 \times G_2}) &= (m_1n_2 + n_1m_2)(n_1n_2 - 1)^4 - 2(n_1n_2 - 1)^3[n_2M_1(G_1) + n_1M_1(G_2) \\ &+ 8m_1m_2] + (n_1n_2 - 1)^2[n_2F(G_1) + n_1F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2)] \\ &+ 4(n_1n_2 - 1)^2[3m_2M_1(G_1) + 3m_1M_1(G_2) + n_1M_2(G_2) + n_2M_2(G_1)] \\ &- 2(n_1n_2 - 1)[n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2][n_2F(G_1) + n_1F(G_2) \\ &+ 6m_2M_1(G_1) + 6m_1M_1(G_2)] + n_2HM_2(G_1) + n_1HM_2(G_2) \\ &+ 3F(G_1)M_1(G_2) + 3F(G_2)M_1(G_1) + m_1[Y(G_2) + 4ReZG_3(G_2)] \\ &+ m_2[Y(G_1) + 4ReZG_3(G_1)] + 4M_1(G_1)M_2(G_2) + 4M_1(G_2)M_2(G_1). \end{aligned}$$

Proof. By (Theorem 3.1) we have

$$\begin{aligned} HM_2(\overline{G}) &= |E(G_1 \times G_2)|(|V(G_1 \times G_2)| - 1)^4 - 2(|V(G_1 \times G_2)| - 1)^3M_1(G_1 \times G_2) \\ &+ (|V(G_1 \times G_2)| - 1)^2F(G_1 \times G_2) + 4(|V(G_1 \times G_2)| - 1)^2M_2(G_1 \times G_2) \\ &- 2(|V(G_1 \times G_2)| - 1)M_1(G_1 \times G_2)F(G_1 \times G_2) + HM_2(G_1 \times G_2). \end{aligned}$$

And $M_1(G_1 \times G_2) = n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2$, $M_2(G_1 \times G_2) = 3m_2M_1(G_1) + 3m_1M_1(G_2) + n_1M_2(G_2) + n_2M_2(G_1)$ given in [20]. $F(G_1 \times G_2) = n_2F(G_1) + n_1F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2)$ given in [23]. $HM_2(G_1 \times G_2) = n_2HM_2(G_1) + n_1HM_2(G_2) + 3F(G_1)M_1(G_2) + 3F(G_2)M_1(G_1) + m_1[Y(G_2) + 4ReZG_3(G_2)] + m_2[Y(G_1) + 4ReZG_3(G_1)] + 4M_1(G_1)M_2(G_2) + 4M_1(G_2)M_2(G_1)$ given in [12] and since $|E(G_1 \times G_2)| = m_1n_2 + n_1m_2, \quad |V(G_1 \times G_2)| = n_1n_2$. Which are complete the proof. □

Definition 3.5. [20],[21] The composition $G_1 \circ G_2$, of two simple and connected graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or $\{u_1 = u_2$ and v_1 is adjacent with $v_2\}$, such that $|V(G_1 \circ G_2)| = n_1n_2$, $|E(G_1 \circ G_2)| = m_1{n_2}^2 + m_2n_1$ and $\delta_{G_1 \circ G_2}(u, v) = n_2\delta_{G_1}(u) + \delta_{G_2}(v)$.

Theorem 3.8. *The second Hyper-Zagreb index of complement* $G_1 \circ G_2$ *is given by:*

$$\begin{aligned} HM_2(\overline{G_1 \circ G_2}) &= (m_1n_2^2 + m_2n_1)(n_1n_2-1)^4 - 2(n_1n_2-1)^3[n_2^3M_1(G_1) \\ &+ n_1M_1(G_2) + 8n_2m_2m_1] + (n_1n_2-1)^2[n_2^4F(G_1) + n_1F(G_2) \\ &+ 6n_2^2m_2M_1(G_1) + 6n_2m_1M_1(G_2)] + 4(n_1n_2-1)^2[n_2^4M_2(G_1) \\ &+ n_1M_2(G_2) + 3n_2^2m_2M_1(G_1) + 2n_2m_1M_1(G_2) + 4m_1m_2^2] \\ &- 2(n_1n_2-1)[n_2^3M_1(G_1) + n_1M_1(G_2) + 8n_2m_2m_1][n_2^4F(G_1) \\ &+ n_1F(G_2) + 6n_2^2m_2M_1(G_1) + 6n_2m_1M_1(G_2)] \\ &+ n_2^6HM_2(G_1) + n_1HM_2(G_2) + n_2^4m_2[Y(G_1) + 4ReZG_3(G_1)] \\ &+ 4n_2m_1ReZG_3(G_2) + 3n_2^2F(G_1)M_1(G_2) + n_2^2M_1(G_1)[F(G_2) \\ &+ 4M_2(G_2)] + m_1M_1^2(G_2) + 4n_2m_2[4n_2m_2M_2(G_1) + M_1(G_1)M_1(G_2)]. \end{aligned}$$

Proof. By using (Theorem 3.1) we have,

$$\begin{aligned} HM_2(\overline{G_1 \circ G_2}) &= |E(G_1 \circ G_2)|(|V(G_1 \circ G_2)|-1)^4 - 2(|V(G_1 \circ G_2)|-1)^3M_1(G_1 \circ G_2) \\ &+ (|V(G_1 \circ G_2)|-1)^2F(G_1 \circ G_2) + 4(|V(G_1 \circ G_2)|-1)^2M_2(G_1 \circ G_2) \\ &- 2(|V(G_1 \circ G_2)|-1)M_1(G_1 \circ G_2)F(G_1 \circ G_2) + HM_2(G_1 \circ G_2), \end{aligned}$$

And by [20], [23], and [12], respectively, we have

$$M_1(G_1 \circ G_2) = n_2^3M_1(G_1) + n_1M_1(G_2) + 8n_2m_2m_1,$$

$$\begin{aligned} M_2(G_1 \circ G_2) &= n_2^4M_2(G_1) + n_1M_2(G_2) + 3n_2^2m_2M_1(G_1) \\ &+ 2n_2m_1M_1(G_2) + 4m_1m_2^2, \end{aligned}$$

$$F(G_1 \circ G_2) = n_2^4F(G_1) + n_1F(G_2) + 6n_2^2m_2M_1(G_1) + 6n_2m_1M_1(G_2),$$

$$\begin{aligned} HM_2(G_1 \circ G_2) &= n_2^6HM_2(G_1) + n_1HM_2(G_2) + n_2^4m_2[Y(G_1) + 4ReZG_3(G_1)] \\ &+ 4n_2m_1ReZG_3(G_2) + 3n_2^2F(G_1)M_1(G_2) + n_2^2M_1(G_1)[F(G_2) \\ &+ 4M_2(G_2)] + m_1M_1^2(G_2) + 4n_2m_2[4n_2m_2M_2(G_1) + M_1(G_1)M_1(G_2)], \end{aligned}$$

And since $|E(G_1 \circ G_2)| = m_1n_2^2 + m_2n_1, \quad |V(G_1 \circ G_2)| = n_1n_2$.
Which are complete the proof. □

Definition 3.6. [20],[21] The disjunction $G_1 \vee G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_1, v_1) is adjacent with (u_2, v_2), whenever $(u_1, u_2) \in E(G_1)$ or $(v_1, v_2) \in E(G_2)$, such that $|V(G_1 \vee G_2)| = n_1n_2$, $|E(G_1 \vee G_2)| = m_1{n_2}^2 + m_2{n_1}^2 - 2m_1m_2$ and $\delta_{G_1 \vee G_2}(u, v) = n_2\delta_{G_1}(u) + n_1\delta_{G_2}(v) - \delta_{G_1}(u)\delta_{G_2}(v)$.

Theorem 3.9. *The second Hyper-Zagreb index of complement $G_1 \vee G_2$ is given by:*

$$\begin{aligned}
HM_2(\overline{G_1 \vee G_2}) &= [m_1n_2^2 + m_2n_1^2 - 2m_1m_2](n_1n_2-1)^4 - 2(n_1n_2-1)^3\big[(n_1n_2^2 \\
&\quad -4m_2n_2)M_1(G_1) + M_1(G_2)M_1(G_1) + (n_2n_1^2 - 4m_1n_1)M_1(G_2) \\
&\quad +8m_1m_2n_1n_2\big] + (n_1n_2-1)^2\big[n_2^4F(G_1) + n_1^4F(G_2) - F(G_1)F(G_2) \\
&\quad +6n_1n_2^2m_2M_1(G_1) + 6n_2n_1^2m_1M_1(G_2) + 3n_2F(G_1)M_1(G_2) \\
&\quad +3n_1F(G_2)M_1(G_1) - 6n_2^2m_2F(G_1) - 6n_1^2m_1F(G_2) - 6n_1n_2M_1(G_1)M_1(G_2)\big] \\
&+ 4(n_1n_2-1)^2\big[((n_1^2-2m_1)^2 - 2n_1^2m_1)M_2(G_2) + ((n_2^2-2m_2)^2 \\
&\quad -2n_2^2m_2)M_2(G_1) + (2n_1^2n_2m_1 - 4m_1^2n_2)M_1(G_2) \\
&\quad +(2n_2^2n_1m_2 - 4m_2^2n_1)M_1(G_1) - n_1n_2M_1(G_2)M_1(G_1) \\
&\quad +2n_2M_2(G_1)M_1(G_2) + 2n_1M_2(G_2)M_1(G_1) - 2M_2(G_2)M_2(G_1) \\
&\quad +4m_2m_1(n_2^2m_1 + n_1^2m_2)\big] - 2(n_1n_2-1)\big[(n_1n_2^2 \\
&\quad -4m_2n_2)M_1(G_1) + M_1(G_2)M_1(G_1) + (n_2n_1^2 - 4m_1n_1)M_1(G_2) \\
&\quad +8m_1m_2n_1n_2\big]\big[n_2^4F(G_1) + n_1^4F(G_2) - F(G_1)F(G_2) \\
&\quad +6n_1n_2^2m_2M_1(G_1) + 6n_2n_1^2m_1M_1(G_2) + 3n_2F(G_1)M_1(G_2) \\
&\quad +3n_1F(G_2)M_1(G_1) - 6n_2^2m_2F(G_1) - 6n_1^2m_1F(G_2) - 6n_1n_2M_1(G_1)M_1(G_2)\big] \\
&+ \big[HM_2(G_2)[n_1(n_1^5 + 16n_1m_1^2 - 10n_1^3m_1 - 8n_1M_2(G_1) \\
&\quad -2n_1F(G_1) + 4ReZG_3(G_1)) + M_1(G_1)(M_1(G_1) + 6n_1^3 - 8n_1m_1)] \\
&\quad +n_2^2F(G_2)M_1(G_1)[n_1^3 - 4n_1m_1 + M_1(G_1)] + HM_2(G_1)[n_2(n_2^5 \\
&\quad +16n_2m_2^2 - 10n_2^3m_2 - 8n_2M_2(G_2) - 2n_2F(G_2) + 4ReZG_3(G_2)) \\
&\quad +M_1(G_2)(M_1(G_2) + 6n_2^3 - 8n_2m_2)] + n_1^2F(G_1)M_1(G_2)[n_2^3 \\
&\quad -4n_2m_2 + M_1(G_2)] + 2n_2ReZG_3(G_2)[n_1^2(2n_1^2m_1 + F(G_1) - 8m_1^2 \\
&\quad +4M_2(G_1)) - M_1(G_1)(M_1(G_1) + 2n_1^3 - 6n_1m_1)] + 4n_2^2M_2(G_2)[4n_1^2m_1^2 \\
&\quad +M_1^2(G_1) - 4n_1m_1M_1(G_1)] + 2n_1ReZG_3(G_1)[n_2^2(2n_2^2m_2 + F(G_2) \\
&\quad -8m_2^2 + 4M_2(G_2)) - M_1(G_2)(M_1(G_2) + 2n_2^3 - 6n_2m_2)] \\
&\quad +4n_1^2M_2(G_1)[4n_2^2m_2^2 + M_1^2(G_2) - 4n_2m_2M_1(G_2)] \\
&\quad +2M_1(G_1)M_1(G_2)[n_1^3(2n_2m_2 - M_1(G_2)) + n_2^3(2n_1m_1 - M_1(G_1))] \\
&\quad +n_1^4m_1M_1^2(G_2) + n_2^4m_2M_1^2(G_1) - 6n_1n_2ReZG_3(G_1)ReZG_3(G_2) \\
&\quad -n_1^2n_2^2[F(G_1)F(G_2) + 8M_2(G_1)M_2(G_2)] - 2HM_2(G_1)HM_2(G_2)\big]
\end{aligned}$$

Proof. By using (Theorem 3.1) we have,

$$\begin{aligned}
HM_2(\overline{G_1 \vee G_2}) &= |E(G_1 \vee G_2)|(|V(G_1 \vee G_2)|-1)^4 - 2(|V(G_1 \vee G_2)|-1)^3M_1(G_1 \vee G_2) \\
&+ (|V(G_1 \vee G_2)|-1)^2F(G_1 \vee G_2) + 4(|V(G_1 \vee G_2)|-1)^2M_2(G_1 \vee G_2) \\
&- 2(|V(G_1 \vee G_2)|-1)M_1(G_1 \vee G_2)F(G_1 \vee G_2) + HM_2(G_1 \vee G_2),
\end{aligned}$$

And by [25],[27],[23],[24] and [12], respectively, we have

$$\begin{aligned}
M_1(G_1 \vee G_2) &= (n_1n_2^2 - 4m_2n_2)M_1(G_1) + M_1(G_2)M_1(G_1) \\
&+ (n_2n_1^2 - 4m_1n_1)M_1(G_2) + 8m_1m_2n_1n_2.
\end{aligned}$$

$$\begin{aligned}
M_2(G_1 \vee G_2) &= ((n_1^2 - 2m_1)^2 - 2n_1^2 m_1)M_2(G_2) + ((n_2^2 - 2m_2)^2 \\
&- 2n_2^2 m_2)M_2(G_1) + (2n_1^2 n_2 m_1 - 4m_1^2 n_2)M_1(G_2) \\
&+ (2n_2^2 n_1 m_2 - 4m_2^2 n_1)M_1(G_1) - n_1 n_2 M_1(G_2)M_1(G_1) \\
&+ 2n_2 M_2(G_1)M_1(G_2) + 2n_1 M_2(G_2)M_1(G_1) - 2M_2(G_2)M_2(G_1) \\
&+ 4m_2 m_1(n_2^2 m_1 + n_1^2 m_2).
\end{aligned}$$

$$\begin{aligned}
F(G_1 \vee G_2) &= n_2^4 F(G_1) + n_1^4 F(G_2) - F(G_1)F(G_2) + 6n_1 n_2^2 m_2 M_1(G_1) \\
&+ 6n_2 n_1^2 m_1 M_1(G_2) + 3n_2 F(G_1)M_1(G_2) + 3n_1 F(G_2)M_1(G_1) \\
&- 6n_2^2 m_2 F(G_1) - 6n_1^2 m_1 F(G_2) - 6n_1 n_2 M_1(G_1)M_1(G_2).
\end{aligned}$$

$$\begin{aligned}
HM_2(G_1 \vee G_2) &= HM_2(G_2)[n_1(n_1^5 + 16n_1 m_1^2 - 10n_1^3 m_1 - 8n_1 M_2(G_1) \\
&- 2n_1 F(G_1) + 4ReZG_3(G_1)) + M_1(G_1)(M_1(G_1) + 6n_1^3 - 8n_1 m_1)] \\
&+ n_2^2 F(G_2)M_1(G_1)[n_1^3 - 4n_1 m_1 + M_1(G_1)] + HM_2(G_1)[n_2(n_2^5 \\
&+ 16n_2 m_2^2 - 10n_2^3 m_2 - 8n_2 M_2(G_2) - 2n_2 F(G_2) + 4ReZG_3(G_2)) \\
&+ M_1(G_2)(M_1(G_2) + 6n_2^3 - 8n_2 m_2)] + n_1^2 F(G_1)M_1(G_2)[n_2^3 \\
&- 4n_2 m_2 + M_1(G_2)] + 2n_2 ReZG_3(G_2)[n_1^2(2n_1^2 m_1 + F(G_1) - 8m_1^2 \\
&+ 4M_2(G_1)) - M_1(G_1)(M_1(G_1) + 2n_1^3 - 6n_1 m_1)] + 4n_2^2 M_2(G_2)[4n_1^2 m_1^2 \\
&+ M_1^2(G_1) - 4n_1 m_1 M_1(G_1)] + 2n_1 ReZG_3(G_1)[n_2^2(2n_2^2 m_2 + F(G_2) \\
&- 8m_2^2 + 4M_2(G_2)) - M_1(G_2)(M_1(G_2) + 2n_2^3 - 6n_2 m_2)] \\
&+ 4n_1^2 M_2(G_1)[4n_2^2 m_2^2 + M_1^2(G_2) - 4n_2 m_2 M_1(G_2)] \\
&+ 2M_1(G_1)M_1(G_2)[n_1^3(2n_2 m_2 - M_1(G_2)) + n_2^3(2n_1 m_1 - M_1(G_1))] \\
&+ n_1^4 m_1 M_1^2(G_2) + n_2^4 m_2 M_1^2(G_1) - 6n_1 n_2 ReZG_3(G_1)ReZG_3(G_2) \\
&- n_1^2 n_2^2[F(G_1)F(G_2) + 8M_2(G_1)M_2(G_2)] - 2HM_2(G_1)HM_2(G_2),
\end{aligned}$$

And since $|E(G_1 \vee G_2)| = m_1 n_2^2 + m_2 n_1^2 - 2m_1 m_2, \quad |V(G_1 \vee G_2)| = n_1 n_2$.
Which are complete the proof. □

Definition 3.7. [20],[21] The symmetric difference $G_1 \oplus G_2$, of two simple and connected graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and $E(G_1 \oplus G_2) = (u_1, u_2)(v_1, v_2)|u_1 v_1 \in E(G_1)$ or $u_2 v_2 \in E(G_2)$ but not both, such that $|V(G_1 \oplus G_2)| = n_1 n_2$, $|E(G_1 \oplus G_2)| = m_1 {n_2}^2 + m_2 {n_1}^2 - 4m_1 m_2$ and $\delta_{G_1 \oplus G_2}(u, v) = n_2 \delta_{G_1}(u) + n_1 \delta_{G_2}(v) - 2\delta_{G_1}(u)\delta_{G_2}(v)$.

Theorem 3.10. *The second Hyper-Zagreb index of complement $G_1 \oplus G_2$ is given by:*

$$\begin{aligned}
HM_2(\overline{G_1 \oplus G_2}) &= [m_1n_2^2 + m_2n_1^2 - 4m_1m_2](n_1n_2-1)^4 - 2(n_1n_2-1)^3\big[(n_1n_2^2 \\
&\quad -8m_2n_2)M_1(G_1) + 4M_1(G_1)M_1(G_2) + (n_2n_1^2 - 8m_1n_1)M_1(G_2) \\
&\quad +8m_1m_2n_1n_2\big] + (n_1n_2-1)^2\big[n_2^4F(G_1) + n_1^4F(G_2) - 8F(G_1)F(G_2) \\
&\quad +6n_1n_2^2m_2M_1(G_1) + 6n_2n_1^2m_1M_1(G_2) + 12n_2F(G_1)M_1(G_2) \\
&\quad +12n_1F(G_2)M_1(G_1) - 12n_2^2m_2F(G_1) - 12n_1^2m_1F(G_2) \\
&\quad -12n_1n_2M_1(G_1)M_1(G_2)\big] + 4(n_1n_2-1)^2\big[((n_1^2-2m_1)^2 - 4n_1^2m_1)M_2(G_2) \\
&\quad +((n_2^2-2m_2)^2 - 4n_2^2m_2)M_2(G_1) + (2n_1^2n_2m_1 - 8m_1^2n_2)M_1(G_2) \\
&\quad +(2n_2^2n_1m_2 - 8m_2^2n_1)M_1(G_1) - 2n_1n_2M_1(G_2)M_1(G_1) \\
&+ \ 8n_2M_2(G_1)M_1(G_2) + 8n_1M_2(G_2)M_1(G_1) - 16M_2(G_2)M_2(G_1) \\
&\quad +4m_2m_1(n_2^2m_1 + n_1^2m_2)\big] - 2(n_1n_2-1)\big[(n_1n_2^2 \\
&\quad -8m_2n_2)M_1(G_1) + 4M_1(G_1)M_1(G_2) + (n_2n_1^2 - 8m_1n_1)M_1(G_2) \\
&\quad +8m_1m_2n_1n_2\big]\big[n_2^4F(G_1) + n_1^4F(G_2) - 8F(G_1)F(G_2) \\
&\quad +6n_1n_2^2m_2M_1(G_1) + 6n_2n_1^2m_1M_1(G_2) + 12n_2F(G_1)M_1(G_2) \\
&\quad +12n_1F(G_2)M_1(G_1) - 12n_2^2m_2F(G_1) - 12n_1^2m_1F(G_2) \\
&\quad -12n_1n_2M_1(G_1)M_1(G_2)\big] + \big[HM_2(G_2)[n_1(n_1^5 + 64n_1m_1^2 - 20n_1^3m_1 \\
&\quad -64n_1M_2(G_1) - 16n_1F(G_1) + 64ReZG_3(G_1)) + M_1(G_1)(16M_1(G_1) \\
&\quad +24n_1^3 - 64n_1m_1)] + n_2^2F(G_2)M_1(G_1)[n_1^3 - 8n_1m_1 + 4M_1(G_1)] \\
&+ \ HM_2(G_1)[n_2(n_2^5 + 64n_2m_2^2 - 20n_2^3m_2 - 64n_2M_2(G_2) - 16n_2F(G_2) \\
&\quad +64ReZG_3(G_2)) + M_1(G_2)(16M_1(G_2) + 24n_2^3 - 64n_2m_2)] \\
&\quad +n_1^2F(G_1)M_1(G_2)[n_2^3 - 8n_2m_2 + 4M_1(G_2)] + 4n_2ReZG_3(G_2) \\
&\quad .[n_1^2(2n_1^2m_1 + 2F(G_1) - 8m_1^2 + 8M_2(G_1)) - M_1(G_1)(4M_1(G_1) \\
&\quad +2n_1^3 - 12n_1m_1)] + 16n_2^2M_2(G_2)[n_1^2m_1^2 + M_1^2(G_1) - 2n_1m_1M_1(G_1)] \\
&\quad +4n_1ReZG_3(G_1)[n_2^2(n_2^2m_2 + 2F(G_2) - 8m_2^2 + 8M_2(G_2)) \\
&\quad -M_1(G_2)(4M_1(G_2) + 2n_2^3 - 12n_2m_2)] + 16n_1^2M_2(G_1)[n_2^2m_2^2 \\
&\quad +M_1^2(G_2) - 2n_2m_2M_1(G_2)] + 4M_1(G_1)M_1(G_2)[n_1^3[n_2m_2 - M_1(G_2)] \\
&\quad +n_2^3(n_1m_1 - M_1(G_1))] + n_1^4m_1M_1^2(G_2) + n_2^4m_2M_1^2(G_1) \\
&\quad -48n_1n_2ReZG_3(G_1)ReZG_3(G_2) - 2n_1^2n_2^2[F(G_1)F(G_2) \\
&\quad +8M_2(G_1)M_2(G_2)] - 64HM_2(G_1)HM_2(G_2)\big].
\end{aligned}$$

Proof. By using (Theorem 3.1) we have,

$$\begin{aligned}
HM_2(\overline{G_1 \oplus G_2}) &= |E(G_1 \oplus G_2)|(|V(G_1 \oplus G_2)|-1)^4 - 2(|V(G_1 \oplus G_2)|-1)^3M_1(G_1 \oplus G_2) \\
&+ (|V(G_1 \oplus G_2)|-1)^2F(G_1 \oplus G_2) + 4(|V(G_1 \oplus G_2)|-1)^2M_2(G_1 \oplus G_2) \\
&- 2(|V(G_1 \oplus G_2)|-1)M_1(G_1 \oplus G_2)F(G_1 \oplus G_2) + HM_2(G_1 \oplus G_2),
\end{aligned}$$

And by [25],[27],[23],[24] and [12], respectively, we have

$$\begin{aligned}
M_1(G_1 \oplus G_2) &= (n_1n_2^2 - 8m_2n_2)M_1(G_1) + 4M_1(G_1)M_1(G_2) \\
&+ (n_2n_1^2 - 8m_1n_1)M_1(G_2) + 8m_1m_2n_1n_2.
\end{aligned}$$

$$
\begin{aligned}
M_2(G_1 \oplus G_2) &= ((n_1^2-2m_1)^2-4n_1^2m_1)M_2(G_2)+((n_2^2-2m_2)^2\\
&- 4n_2^2m_2)M_2(G_1)+(2n_1^2n_2m_1-8m_1^2n_2)M_1(G_2)\\
&+ (2n_2^2n_1m_2-8m_2^2n_1)M_1(G_1)-2n_1n_2M_1(G_2)M_1(G_1)\\
&+ 8n_2M_2(G_1)M_1(G_2)+8n_1M_2(G_2)M_1(G_1)-16M_2(G_2)M_2(G_1)\\
&+ 4m_2m_1(n_2^2m_1+n_1^2m_2).
\end{aligned}
$$

$$
\begin{aligned}
F(G_1 \oplus G_2) &= n_2^4F(G_1)+n_1^4F(G_2)-8F(G_1)F(G_2)+6n_1n_2^2m_2M_1(G_1)\\
&+ 6n_2n_1^2m_1M_1(G_2)+12n_2F(G_1)M_1(G_2)+12n_1F(G_2)M_1(G_1)\\
&- 12n_2^2m_2F(G_1)-12n_1^2m_1F(G_2)-12n_1n_2M_1(G_1)M_1(G_2).
\end{aligned}
$$

$$
\begin{aligned}
HM_2(G_1 \oplus G_2) &= HM_2(G_2)[n_1(n_1^5+64n_1m_1^2-20n_1^3m_1-64n_1M_2(G_1)\\
&- 16n_1F(G_1)+64ReZG_3(G_1))+M_1(G_1)(16M_1(G_1)\\
&+ 24n_1^3-64n_1m_1)]+n_2^2F(G_2)M_1(G_1)[n_1^3-8n_1m_1+4M_1(G_1)]\\
&+ HM_2(G_1)[n_2(n_2^5+64n_2m_2^2-20n_2^3m_2-64n_2M_2(G_2)-16n_2F(G_2)\\
&+ 64ReZG_3(G_2))+M_1(G_2)(16M_1(G_2)+24n_2^3-64n_2m_2)]\\
&+ n_1^2F(G_1)M_1(G_2)[n_2^3-8n_2m_2+4M_1(G_2)]+4n_2ReZG_3(G_2)\\
&. \; [n_1^2(2n_1^2m_1+2F(G_1)-8m_1^2+8M_2(G_1))-M_1(G_1)(4M_1(G_1)\\
&+ 2n_1^3-12n_1m_1)]+16n_2^2M_2(G_2)[n_1^2m_1^2+M_1^2(G_1)-2n_1m_1M_1(G_1)]\\
&+ 4n_1ReZG_3(G_1)[n_2^2(n_2^2m_2+2F(G_2)-8m_2^2+8M_2(G_2))\\
&- M_1(G_2)(4M_1(G_2)+2n_2^3-12n_2m_2)]+16n_1^2M_2(G_1)[n_2^2m_2^2\\
&+ M_1^2(G_2)-2n_2m_2M_1(G_2)]+4M_1(G_1)M_1(G_2)[n_1^3[n_2m_2-M_1(G_2)]\\
&+ n_2^3(n_1m_1-M_1(G_1))]+n_1^4m_1M_1^2(G_2)+n_2^4m_2M_1^2(G_1)\\
&- 48n_1n_2ReZG_3(G_1)ReZG_3(G_2)-2n_1^2n_2^2[F(G_1)F(G_2)\\
&+ 8M_2(G_1)M_2(G_2)]-64HM_2(G_1)HM_2(G_2),
\end{aligned}
$$

And since $|E(G_1 \oplus G_2)| = m_1n_2^2+m_2n_1^2-4m_1m_2$, $\quad |V(G_1 \oplus G_2)| = n_1n_2$. Which are complete the proof. □

3.2 Second Hyper-Zagreb index of complement V-Phenylenic Nanotubes $VPHX[q,p](\forall p,q \in \mathbb{N}-\{1\})$

Corollary 3.11. *The second Hyper-Zagreb index of complement $VPHX[q,p]$ nanotube (Figs. 1, 2) is given by*

$$
\begin{aligned}
HM_2(\overline{VPHX[q,p]}) &= (9pq-q)(6pq-1)^4-2(6pq-1)^3[54pq-10q]\\
&+ (6pq-1)^2[162pq-38q]+4(6pq-1)^2[81pq+3q]\\
&- 8q(6pq-1)[54pq-10q](162pq-38q)+9q(81p-29).
\end{aligned}
$$

Table 1. The edge and vertex partitions of $VPHX[q,p]$ nanotubes

Edge partition	$E_5 = E_6^*$	$E_6 = E_9^*$	Vertex partition	V_2	V_3
Cardinality	$4q$	$9pq-5q$	*Cardinality*	$q+q$	$6pq-2q$

The edge set of $VPHX[q,p]$ is divided into two edge partitions based on the sum of degrees of the end vertices as:

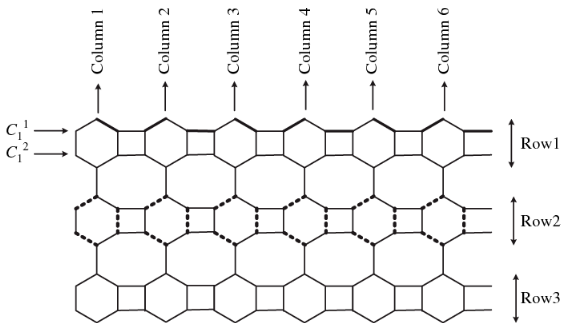

Fig. 1. $VPHX[3,6]$ **nanotube**

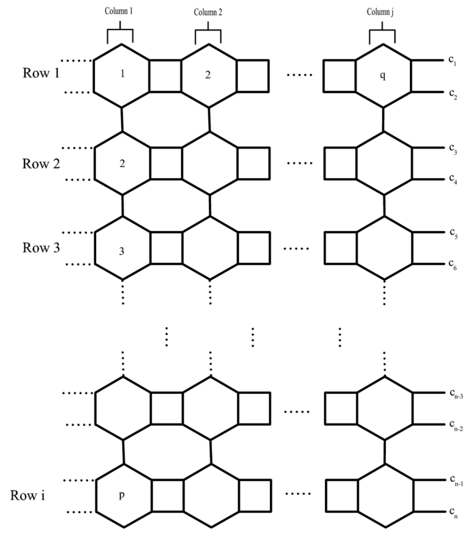

Fig. 2. The molecular graph of $VPHX[q,p]$ **nanotube.**

$$E_5(VPHX[q,p]) = E_6^* = \{e = uv \in E(VPHX[q,p]) : \delta(u) = 2, \delta(v) = 3\},$$

$$E_6(VPHX[q,p]) = E_9^* = \{e = uv \in E(VPHX[q,p]) : \delta(u) = 3, \delta(v) = 3\},$$

Proof. By using (Theorem 3.1) we have

$$\begin{aligned} HM_2(\overline{VPHX[q,p]}) &= |E(VPHX[q,p])|(|V(VPHX[q,p])| - 1)^4 \\ &- 2(|V(VPHX[q,p])| - 1)^3 M_1(VPHX[q,p]) \\ &+ (|V(VPHX[q,p])| - 1)^2 F(VPHX[q,p]) \\ &+ 4(|V(VPHX[q,p])| - 1)^2 M_2(VPHX[q,p]) \\ &- 2(|V(VPHX[q,p])| - 1) M_1(VPHX[q,p]) F(VPHX[q,p]) \\ &+ HM_2(VPHX[q,p]). \end{aligned}$$

And $M_1(VPHX[q,p]) = 54pq - 10q$, $M_2(VPHX[q,p]) = 81pq + 3q$, and $HM_2(VPHX[q,p]) = 9q(81p - 29)$, given in (Theorem 2.1)above. And since $F(VPHX[q,p]) = 162pq - 38q$ given in [26] and the partitions of the vertex set and edge set of $(VPHX[q,p])$ nanotubes are given in (Table 1) respectively [11],[19].

$$\sum |V(VPHX[q,p])| = 6pq, \qquad \sum |E(VPHX[q,p])| = 9pq - q$$

Which are complete the proof. □

3.3 Second Hyper-Zagreb index of complement V-Phenylenic Nanotorus $VPHY[m,n] (\forall m, n \in \mathbb{N} - \{1\})$

Corollary 3.12. *The second Hyper-Zagreb index of complement* $VPHY[m,n]$ *nanotorus (Fig. 3) is given by*

$$\begin{aligned} HM_2(\overline{VPHY[m,n]}) &= 9mn(6mn-1)^4 - 108mn(6mn-1)^3 + 486mn(6mn-1)^2 \\ &\quad -17496m^2n^2(6mn-1) + 729mn. \end{aligned}$$

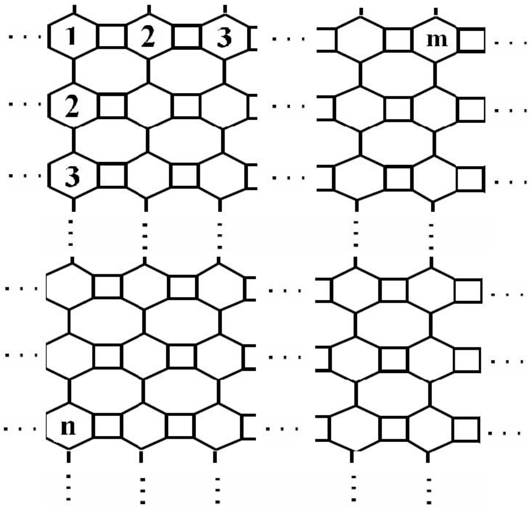

Fig. 3. The molecular graph of $VPHY[m,n]$ nanotorus

Table 2. The edge and vertex partitions of $VPHY[m,n]$ nanotorus

Edge partition	$E_6 = E_9^*$	Vertex partition	V_3
Cardinality	$9mn$	*Cardinality*	$6mn$

The edge set of $VPHY[m,n]$ have only one type of edges:

$$E_6(VPHY[m,n]) = E_9^* = \{e = uv \in E(VPHY[m,n]) : \delta(u) = 3, \delta(v) = 3\},$$

Proof. By using (Theorem 3.1) we have

$$\begin{aligned} HM_2(\overline{VPHY[m,n]}) &= |E(VPHY[m,n])|(|V(VPHY[m,n])| - 1)^4 \\ &- 2(|V(VPHY[m,n])| - 1)^3 M_1(VPHY[m,n]) \\ &+ (|V(VPHY[m,n])| - 1)^2 F(VPHY[m,n]) \\ &+ 4(|V(VPHY[m,n])| - 1)^2 M_2(VPHY[m,n]) \\ &- 2(|V(VPHY[m,n])| - 1) M_1(VPHY[m,n]) F(VPHY[m,n]) \\ &+ HM_2(VPHY[m,n]). \end{aligned}$$

And $M_1(VPHY[m,n]) = 54mn$, $M_2(VPHY[m,n]) = 81mn$, and $HM_2(VPHY[m,n]) = 729mn$, given in (Theorem 2.2)above. And since $F(VPHY[m,n]) = 162mn$ given in [26] and the partitions of the vertex set and edge set of $(VPHY[m,n])$ nanotubes are given in (Table 2) [11],[19].

$$\sum |V(VPHY[m,n])| = 6mn, \qquad \sum |E(VPHY[m,n])| = 9mn$$

Which are complete the proof. □

3.4 Second Hyper-Zagreb index of complement Titania nanotubes and nanotorus

Titania nanotubes are considered one of the most studied compounds in materials science. Owing to some outstanding properties, it is used, for instance, in photocatalysis, dye-sensitized solar cells, and biomedical devices [28],[29],[30]. In the following subsection, we study the second Hyper-Zagreb of molecular complement graph Titania nanotubes and molecular complement graph of nanotorus.

Corollary 3.13. *The second Hyper-Zagreb index of complement $TiO_2[n,m]$ nanotube (Fig. 4) is given by*

$$\begin{aligned} HM_2(\overline{TiO_2[n,m]}) &= [10mn+8n]\big[6mn+6n-1\big]^4 \\ &- 2(76mn+48n)\big[6mn+6n-1\big]^3+(840mn+408n)\big[6mn+6n-1\big]^2 \\ &+ 1930mn+362n-2\big[6mn+6n-1\big](76mn+48n)(320mn+160n). \end{aligned}$$

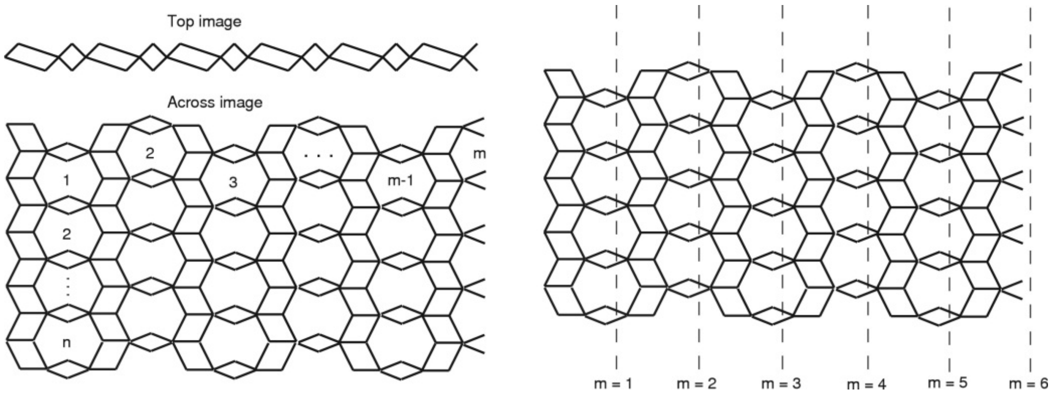

Fig. 4. The molecular graph of $TiO_2[n,m]$ nanotube

Proof. By using (Theorem 3.1) we have

$$\begin{aligned} HM_2(\overline{TiO_2[n,m]}) &= |E(TiO_2)|(|V(TiO_2)|-1)^4-2(|V(TiO_2)|-1)^3M_1(TiO_2) \\ &+ (|V(TiO_2)|-1)^2F(TiO_2)+4(|V(TiO_2)|-1)^2M_2(TiO_2) \\ &- 2(|V(TiO_2)|-1)M_1(TiO_2)F(TiO_2)+HM_2(TiO_2). \end{aligned}$$

And since $M_1(TiO_2[n,m]) = 76mn+48n$, $M_2(TiO_2[n,m]) = 130mn+62n$. $F(TiO_2[n,m]) = 320mn+160n$ given in [28],[29]. $HM_2(TiO_2[n,m]) = 1930mn+362n$ given in [30]. and the partitions of the vertex set and edge set $V(TiO_2), E(TiO_2)$, of $TiO_2[n,m]$ nanotubes are given in (Tables 3, 4), respectively. We have $\sum|V(TiO_2[n,m])| = 6mn+6n$, $\sum|E(TiO_2[n,m])| = 10mn+8n$

Table 3. The vertex partition of $TiO_2[n,m]$ nanotubes

Vertex partition	v_2	v_3	v_4	v_5
Cardinality	$2mn+4n$	$2mn$	$2n$	$2mn$

Table 4. The edge partition of $TiO_2[n,m]$ nanotubes

Edge partition	$E_6=E_8^*$	$E_7=E_{10}^*\cup E_{12}^*$	$E_8=E_{15}^*$	E_{12}^*	E_{10}^*
Cardinality	$6n$	$4mn+4n$	$6mn-2n$	$4mn+2n$	$2n$

$$\begin{aligned}
HM_2(\overline{TiO_2[n,m]}) &= \sum |E(TiO_2[n,m])|\big[\sum |V(TiO_2[n,m])| - 1\big]^4 \\
&- 2M_1(TiO_2)\big[\sum |V(TiO_2[n,m])| - 1\big]^3 \\
&+ F(TiO_2)\big[\sum |V(TiO_2[n,m])| - 1\big]^2 \\
&+ 4M_2(TiO_2)\big[\sum |V(TiO_2[n,m])| - 1\big]^2 \\
&- 2\big[\sum |V(TiO_2[n,m])| - 1\big]M_1(TiO_2)F(TiO_2) + HM_2(TiO_2) \\
&= [|E_8^*| + |E_{10}^* \cup E_{12}^*| + |E_{15}^*|]\big[6mn + 6n - 1\big]^4 \\
&- 2(76mn + 48n)\big[6mn + 6n - 1\big]^3 + (320mn + 160n)\big[6mn + 6n - 1\big]^2 \\
&+ 4(130mn + 62n)\big[6mn + 6n - 1\big]^2 + 1930mn + 362n \\
&- 2\big[6mn + 6n - 1\big](76mn + 48n)(320mn + 160n) \\
&= [10mn + 8n]\big[6mn + 6n - 1\big]^4 \\
&- 2(76mn + 48n)\big[6mn + 6n - 1\big]^3 + (840mn + 408n)\big[6mn + 6n - 1\big]^2 \\
&+ 1930mn + 362n - 2\big[6mn + 6n - 1\big](76mn + 48n)(320mn + 160n).
\end{aligned}$$

□

Corollary 3.14. *[32] Let $T = T[p,q]$ be the molecular graph of a nanotorus such that $|V(T)| = pq$, $|E(T)| = \frac{3}{2}pq$ (Fig. 5). Then,*

a. $ReZG_3(T[p,q]) = 81pq$.

b. $HM_2(T[p,q]) = \frac{243}{2}pq$.

c. $HM_2(P_n \times T) = pq[\frac{2837}{2}n - 1831]$.

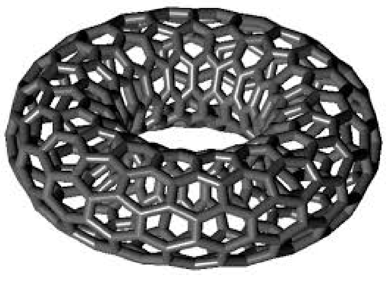

Fig. 5. Molecular graph of a nanotorus

Corollary 3.15. *Let $T = T[p,q]$ be the molecular graph of a nanotorus such that $|V(T)| = pq$, $|E(T)| = \frac{3}{2}pq$ (Fig. 5) Then,*

a. $HM_2(\overline{T[p,q]}) = \dfrac{3}{2}pq(pq-1)^4 - 18pq(pq-1)^3 + 81pq(pq-1)^2 - 486p^2q^2(pq-1) + \dfrac{243}{2}pq.$

b.
$$\begin{aligned}
HM_2(\overline{P_n \times T}) &= pq\big[(npq-1)^3[\frac{5}{2}n^2pq - npq - \frac{105}{2}n + 37] + (npq-1)^2[375n - 370] \\
&- 2pq(25n-18)(npq-1)(125n-122) + \frac{2837}{2}n - 1831\big].
\end{aligned}$$

Proof. To prove (a), by using (Theorem 3.1) we have

$$\begin{aligned} HM_2(\overline{T[p,q]}) &= |E(T[p,q])|(|V(T[p,q])|-1)^4 - 2(|V(T[p,q])|-1)^3 M_1(T[p,q]) \\ &+ (|V(T[p,q])|-1)^2 F(T[p,q]) + 4(|V(T[p,q])|-1)^2 M_2(T[p,q]) \\ &- 2(|V(T[p,q])|-1) M_1(T[p,q]) F(T[p,q]) + HM_2(T[p,q]). \end{aligned}$$

And since $M_1(T) = 9pq$, $M_2(T) = \frac{27}{2}pq$ given in [15]. $F(T) = 27pq$ given in [31]. $HM(T[p,q]) = 54pq$ given in [24]. and $HM_2(T[p,q]) = \frac{243}{2}pq$ given in (Corollary 3.14 (b)) above. Then,

$$\begin{aligned} HM_2(\overline{T[p,q]}) &= |E(T[p,q])|(|V(T[p,q])|-1)^4 - 2(|V(T[p,q])|-1)^3 M_1(T[p,q]) \\ &+ (|V(T[p,q])|-1)^2 F(T[p,q]) + 4(|V(T[p,q])|-1)^2 M_2(T[p,q]) \\ &- 2(|V(T[p,q])|-1) M_1(T[p,q]) F(T[p,q]) + HM_2(T[p,q]) \\ &= \frac{3}{2}pq(pq-1)^4 - 18pq(pq-1)^3 \\ &+ 81pq(pq-1)^2 - 486p^2q^2(pq-1) + \frac{243}{2}pq \end{aligned}$$

To prove (b), since $M_1(P_n \times T) = pq(25n-18)$, $M_2(P_n \times T) = \frac{1}{2}pq(125n-124)$, given in [23]. and since

$$\begin{aligned} F(P_n \times T) &= |V(T)|F(P_n) + |V(P_n)|F(T) + 6|E(T)|M_1(P_n) + 6|E(P_n)|M_1(T) \\ &= 2pq(4n-7) + 27npq + 18pq(2n-3) + 54pq(n-1) \\ &= pq[125n-122]. \end{aligned}$$

By (Corollary 3.14 (c)) $HM_2(P_n \times T) = pq[\frac{2837}{2}n - 1831]$ and since $|E(P_n \times T)| = (n-1)pq + \frac{3}{2}npq = pq(\frac{5}{2}n-1)$, $|V(P_n \times T)| = npq$, and by using (Theorem 3.1) we get

$$\begin{aligned} HM_2(\overline{P_n \times T}) &= |E(P_n \times T)|(|V(P_n \times T)|-1)^4 - 2(|V(P_n \times T)|-1)^3 M_1(P_n \times T) \\ &+ (|V(P_n \times T)|-1)^2 F(P_n \times T) + 4(|V(P_n \times T)|-1)^2 M_2(P_n \times T) \\ &- 2(|V(P_n \times T)|-1) M_1(P_n \times T) F(P_n \times T) + HM_2(P_n \times T) \\ &= pq(\frac{5}{2}n-1)(npq-1)^4 - 2pq(npq-1)^3(25n-18) \\ &+ pq[125n-122](npq-1)^2 + 2pq(npq-1)^2(125n-124) \\ &- 2p^2q^2(25n-18)(npq-1)(125n-122) + pq[\frac{2837}{2}n - 1831] \\ &= pq\big[(npq-1)^3[\frac{5}{2}n^2pq - npq - \frac{105}{2}n + 37] + (npq-1)^2[375n-370] \\ &- 2pq(25n-18)(npq-1)(125n-122) + \frac{2837}{2}n - 1831\big]. \end{aligned}$$

□

4 Conclusions

This article has presented explicit formulas of some basic mathematical operation for the second Hyper Zagreb index of complement graph operations. Moreover, we computed the second hyper-Zagreb index for some certain important physicochemical structures such as molecular complement graphs of V-Phenylenic Nanotube $VPHX[q,p]$, V-Phenylenic Nanotorus $VPHY[m,n]$ and Titania Nanotubes TiO_2.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Gutman I, Trinajstić N. Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, Chem. Phys. Lett. 1972;17:535-538.

[2] Ashrafi A, Došlić T, Hamzeh A. The Zagreb coindices of graph operations, Discret. Appl.Math. 2010;158:1571-1578.

[3] Shirdel G, Rezapour H, Sayadi A. The hyper-Zagreb index of graph operations, Iranian J. Math. Chem. 2013;4(2):213-220.

[4] Veylaki M, Nikmehr M, Tavallaee H. The third and hyper-Zagreb coindices of some graph operations, J. Appl. Math. Comput. 2016;50:315-325.

[5] Furtula B, Gutman I, A forgotten topological index, J. Math. Chem. 2015;53(4):1184-1190.

[6] Furtula B, Gutman I, et al. On an old/new degree-based topological index, Bull. Acad. Serbe Sci. Arts (Cl. Sci. Math. Natur.) 2015;40:19-31.

[7] Ranjini P, Lokesha V, Usha A, Relation between phenylene and hexagonal squeeze using harmonic index, International Journal of Graph Theory. 2013;1:116-121.

[8] Alameri A, Al-Naggar N, Al-Rumaima M, Alsharafi M. Y-index of some graph operations, International Journal of Applied Engineering Research (IJAER). 2020;15(2):173-179.

[9] Alameri A, Al-Rumaima M, Almazah M. Y-coindex of graph operations and its applications of molecular descriptors, Journal of Molecular Structure. 2020;1221:128754.

[10] Wei G, Farahani M, Siddiqui M. On the First and Second Zagreb and First and Second Hyper-Zagreb Indices of Carbon Nanocones $CNC_k[n]$, Journal of Computational and Theoretical Nanoscience. 2016;13(10):7475-7482.

[11] Farahani M, Rajesh Kanna M, Pradeep Kumar R. On the Hyper-Zagreb indices of Nano-structures, Asian Academic Research Journal of Multidisciplinary. 2016;3(1):115-123.

[12] Modabish A, Alameri A, Gumaan M, Alsharafi M. The second hyper-Zagreb index of graph operations, J. Math. Comput. Sci. 2021;11(2):1455-1469.

[13] Alsharafi M, Shubatah M, Alameri A. The hyper-Zagreb index of some complement graphs, Advances in Mathematics: Scientific Journal. 2020;9(6):3631-3642.

[14] Alameri A, Shubatah M, Alsharafi M. Zagreb indices, Hyper Zagreb indices and Redefined Zagreb indices of conical graph, Advances in Mathematics: Scientific Journal. 2020;9(6):3631-3642.

[15] Alsharafi M, Shubatah M, Alameri A. On the Hyper-Zagreb coindex of some Graphs, J. Math. Comput. Sci. 2020;10(5):1875-1890.

[16] Bondy J, Murty U. Graph Theory With Applications. New York: Amsterdam Oxford; 1976.

[17] Bondy J, Murty U. Graph Theory. USA: Springer; 2008.

[18] Alameri A, Alsharafi M. Topological indices types in graphs and their applications, Generis Publishing; 2021.
ISBN: 9798722479891.

[19] Farahani M, Kumar R, Kanna M. First and second zagreb indices and polynomials of V-phenylenic nanotubes and nanotori, Int J Pharm Sci Res. 2017;8(1):330-33.

[20] Khalifeh M, Yousefi-Azari H, Ashrafi A. The first and second Zagreb indices of some graph operations, Discrete applied mathematics. 2009;157(4):804-811.

[21] Behmaram A, Yousefi-Azari H, Ashrafi A. Some New Results on Distance-Based Polynomials, Commun. Math. Comput. Chem. 2011;65:39-50.

[22] Kiruthika K. Zagreb indices and Zagreb coindices of some graph operations, International Journal of Advanced Research in Engineering and Technology. 2016;7(3):25-41.

[23] Nilanjan De, Abu Nayeem S, Pal A. F-Index of some graph operations, Discrete Mathematics, Algorithms and Applications. 2016;8(2)1650025:17.

[24] Alsharafi M, Shubatah M. On the Hyper-Zagreb index of some Graph Binary Operations, Asian Research Journal of Mathematics. 2020;16(4):12-24.

[25] Alsharafi M, Shubatah M, Alameri A. The First and Second Zagreb Index of Complement Graph and Its Applications of Molecular Graph, Asian Journal of Probability and Statistics. 2020;8(3):15-30.

[26] Alsharafi M, Alameri A. The F-index and coindex of V-Phenylenic Nanotubes and Nanotorus and their molecular complement graphs, Nanosystems Physics Chemistry Mathematics. 2021;12(3):263-270.

[27] Alameri A, Alsharafi M, Esmail A, Gumaan M. A note on Topological indices and coindices of disjunction and symmetric difference of graphs, Discrete Applied Mathematics. 2021;304:230-235.

[28] Nilanjan De. On Molecular Topological Properties of TiO_2 Nanotubes, Hindawi Publishing Corporation Journal of Nanoscience; 2016. Article ID 1028031, 5 pages.

[29] Malik M, Imran M. On Multiple Zagreb Indices of TiO_2 Nanotubes, Acta Chim. Slov. 2015;62:973-976.

[30] Alameri A. Second Hyper-Zagreb Index of Titania Nanotubes and Their Applications, Open Access Journal. 2021;9:9567-9571.

[31] Aram H, Dehgardi N. Reformulated F-index of graph operations, Communications in Combinatorics and Optimization. 2017;1(2):87-98.

[32] Ayache A, Alameri A, Alsharafi M, Ahmed H. The Second Hyper-Zagreb Coindex of Chemical Graphs and Some Applications, Journal of Chemistry; 2021. Article ID 3687533, 8 pages.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/76935