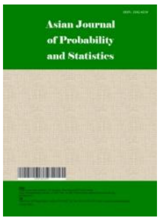

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Kumaraswamy-Janardan Distribution: A Generalized Janardan Distribution with Application to Real Data

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The quest to improve on flexibility of probability distributions motivated this research. Four-parameter Janardan generalized distribution known as Kumaraswamy-Janardan distribution is proposed through method of parameterization and studied. The probability density function, cumulative density function, survival rate function as well as hazard rate function of the distribution are established. Statistical properties such as moments, moment generating function as well as maximum likelihood of the model are discussed. The parameters are estimated using the simulated annealing optimization algorithm. Flexibility of the model in comparison with the baseline model as well as other competing sub-models is verified using Akaike Information Criteria (AIC). The model is tested with real data and is proven to be more flexible in fitting real data than any of its sub-models considered.

Keywords: Janardan; Kumaraswamy; akaike information criteria; parameterization; moment generating function.

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1 Introduction

Classical probability distributions form the basis for many post-classical distributions of which Janardan distribution of no exception. Post-classical probability distributions are developed to be more flexible in fitting lifetime data. One key distribution which has evolved over the years and remains relevant is the exponential distribution. The relevance of the exponential distribution has been discussed and applied to many fields of knowledge including, but not limited to, actuarial, medical, engineering, quality control and demography. The assumptions of exponential distribution have been questioned since 1966 and its hazard rate function problematically remains monotonic [1]. It is conspicuous that the struggle to improve on exponential distribution dates back several decades. Hence it is unsurprising that Shanker's 2013 Janardan distribution is a major enhancement of exponential distribution. The quest to improve on distributions by introducing higher parameter distributions is widespread in the field of probability. Scholars [2,3,4,5] improved upon one distribution or another through parameterization.

Janardan distribution [6] has gained some attention in scholarly literature on probability hence improving upon the relevance as well as usability of the distribution. Shanker et al. [7] introduced a more general form of the Janardan distribution by merging Poisson and Janardan to arrive at the Discrete Poisson Janardan Distribution (PJD) in which Sankaran's [8] discrete Poisson Lindley Distribution (PLD) was a special case. Bashir, S. and Rasul, M. [9] introduced Size-Biased Janardan Distribution, of which A Size-Biased Lindley Distribution is a special case. In their publication, they conclude that "Janardan Distribution is one of the important distributions for lifetime model and it has many applications in real life data". Al-Omari et al. [10] introduced a generalization of Janardan distribution known as Transmuted Janardan distribution (TJD). The Janardan probability distribution has been applied to many lifetime datasets and proven to be more flexible, with better fits, relative to Lindley and exponential distributions.

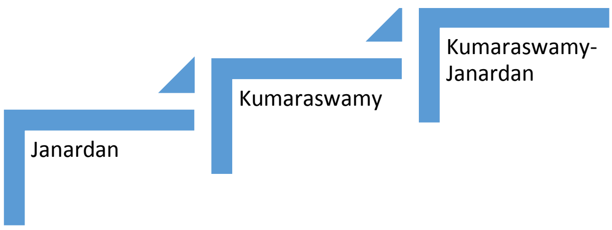
However, Shekari [11] introduced "The Compound Class of Janardan–Power Series Distributions", in a bid to improve the Janardan distribution. Hence this study is proposing another generalized Janardan distribution, determine its suitability and superiority to competing distributions. This is achieved using the Kumaraswamy generator.

In 1980, Kumaraswamy introduced the Kumaraswamy distribution as a better substitute for beta distribution [12]. This was introduced when beta distribution failed to model hydrological processes. Kumaraswamy compared his distribution to some widely used distributions at the time and found his distribution to provide a better fit, after running the considered distributions on hydrological data.

Among the competing distributions are log normal distribution, beta distribution and normal distribution [13]. This two-parameter distribution (Kumaraswamy) has been applied, by many scholars, in modeling test scores, height data, temperature of atmosphere and many more [12, 14]. Since its introduction, many scholars [15,16,17] have presented Kumaraswamy (in their research works) as a better distribution than its computing distributions. Hence, Kumaraswamy is the obvious transformed transformer generator for most distributions that have need for modification to enable them to closely model current lifetime data in this data age. That is the researchers' motivation to choose the Kumaraswamy the generator to improve on the Janardan distribution.

2 Methodology

The proposed Kumaraswamy-Janardan (KJ) distribution is developed using the method of parameterization.



In developing Kumaraswamy-Janardan distribution, the Janardan distribution is substituted into the Kumaraswamy generator using function of functions application to produce the proposed distribution known as Kumaraswamy-Janardan. The parameter estimates are obtained using maximum likelihood estimation, with simulated annealing, in R. Flexibility of the model in comparison with the baseline model as well as other competing sub-models is verified using Akaike Information Criteria (AIC).

3 Theoretical Findings of Kumaraswamy-Janardan Distribution

In this section, we present four-parameter distribution named as Kumaraswamy-Janardan (KJ) distribution. Cumulative density function, probability density function, hazard rate, survival rate functions as well as pictorial presentations of the pdf are established in this section. Also, maximum likelihood estimator as well as moment about the origin are established for KJ.

3.1 Mathematical Derivations of Kumaraswamy-Janardan Distribution

From literature, Shanker et al. [6] established Cumulative density function and density function of Janardan distribution respectively as:

$$G(x,\varphi,\rho) = 1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x};\ x,\varphi,\rho\ > 0 \tag{1}$$

$$g(x,\varphi,\rho\ ) = \frac{\rho^2}{\varphi(\rho+\varphi^2)}.(1+\varphi x)\, e^{-\frac{\rho}{\varphi}x}\ \ ;\ x,\varphi,\rho\ > 0 \tag{2}$$

Also, Kumaraswamy, P. [13] established the cumulative and probability density functions of Kumaraswamy distribution respectively as:

$$F(x) = 1 -\ (1 -\ x^{\alpha})^{\beta} \tag{3}$$

And

$$f(x) = \alpha\beta x^{\alpha-1}(1 -\ x^{\alpha})^{\beta-1} \tag{4}$$

Kumaraswamy distribution was later established as Kumaraswamy generalized by Cordeiro and Castro (2011) with cdf and pdf respectively as:

$$F(x) = 1 -\ (1 -\ G(x)^{\alpha})^{\beta};\ \ 0 < x < 1;\ \alpha,\beta > 0 \tag{5}$$

In order to establish the cumulative density function for Kumaraswamy-Janardan distribution, we employed function of functions approach by substituting (1) into (5).

$$F(x) = \begin{cases} 1 - \left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta} & ;\ x,\varphi,\rho,\alpha,\beta > 0 \\ 0; & elsewhere \end{cases}$$
(6)

Where $\varphi\ is\ "cale"\ parameter\ and\ s\square ape\ parameters\ being\ \rho,\alpha\ and\ \beta$

Differentiating (6);

$$f(x) = \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\left(\frac{\rho x}{\varphi} - 1\right) e^{-\frac{\rho}{\varphi}x}\right] \times \left[\left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right] \times$$
$$\left[\left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] \tag{7}$$

Having established cdf (6) and pdf (7) of Kumaraswamy-Janardan (KJ) probability distribution, the study went further to derive the survival and hazard functions .

Survival function, $R(x)$, by definition is given as

$$R(x) = 1 - F(x)$$

This implies that the survival function for Kumaraswamy Janardan distribution is derived as:

$$R(x) = 1 - \left[1 - \left(1 - \left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta}\right] \tag{8}$$

Hazard function, $H(x)$, by definition is given as:

$$h(x) = \frac{f(x)}{R(x)}$$

This implies that the Hazard rate function of Kumaraswamy-Janardan distribution is derived as:

$$h(x) = \left[\left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\left(\frac{\rho x}{\varphi} - 1\right) e^{-\frac{\rho}{\varphi}x}\right] \times \left[\left(1 - \frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] \times\right.$$

1− 1− φρ+φ2+ρφ2x φρ+φ2 e−ρφxαβ−1 ÷1− 1− 1− 1− φρ+φ2+ρφ2x φρ+φ2 e−ρφxαβ (9)

It is important to establish hazard rate function of KJ distribution because it provides foundation for planning insurance and safety of a system in a wide variety of applications [18]

The KJ distribution is flexible noticing that the distribution at various parameter values exhibits several renowned distributions as sub-models. For instance;

a) When $\alpha = 1$ we obtain three-parameter Exponentiated Janardan distribution with CDF as:

$$F(x) = 1 - \left(- \frac{\varphi(\rho + \varphi^2) + \rho\varphi^2 x}{\varphi(\rho + \varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\beta}$$

b) When $\alpha = \beta = 1$ we obtain two-parameter Janardan distribution with CDF as:

$$F(x) = 1 - \frac{\varphi(\rho + \varphi^2) + \rho\varphi^2 x}{\varphi(\rho + \varphi^2)}\ e^{-\frac{\rho}{\varphi}x}$$

c) When $\alpha = \beta = \varphi = 1$ we obtain one-parameter Lindley distribution with CDF as:

$$F(x) = 1 - \frac{(\rho + 1)\rho x}{(\rho + 1)}\ e^{-\rho x}$$

d) When $\alpha = \beta = \rho = 1$ we obtain gamma distribution with CDF as:

$$F(\ \varphi) = - \frac{\varphi}{(1 + \varphi^2)}\ x e^{-\frac{1}{\varphi}x}$$

3.2 Visualization of Kumaraswamy-Janardan Distribution

This section presents pictorial analysis of the proposed distribution.
From Fig. 1, it is conspicuous that Kumaraswamy-Janardan distribution is a unimodal probability distribution. Depending on parameter values, the distribution depicts flexibility in modeling datasets which are right-skewed or nearly symmetric.

Graphical illustration of effect of the various parameters on the distribution is presented in Fig. 2 through Fig. 5. Careful observation of the behavior of the figures reveal that φ is scale parameter while , ρ and β are the shape parameters. φ being scale parameter controls the variability and scalability in the dataset. The roles of these parameters are demonstrated pictorially.

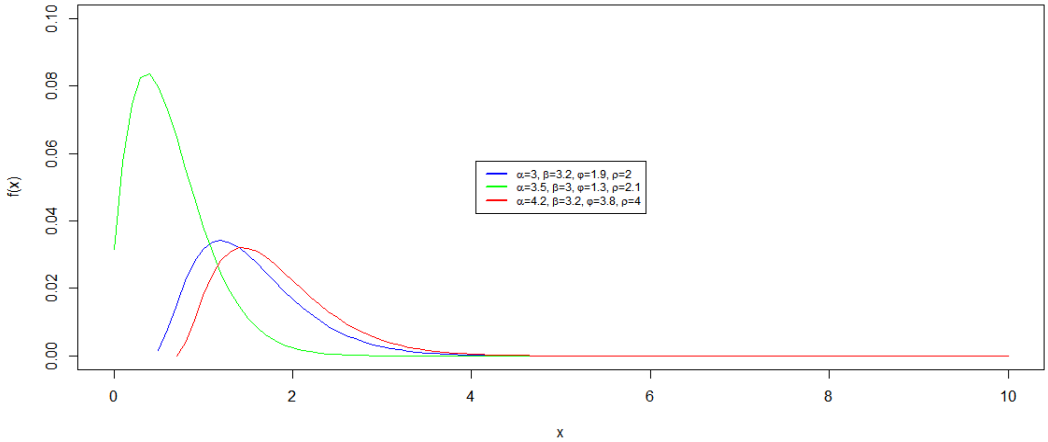


Fig. 1. Behaviour of pdf of KJ for some parameters

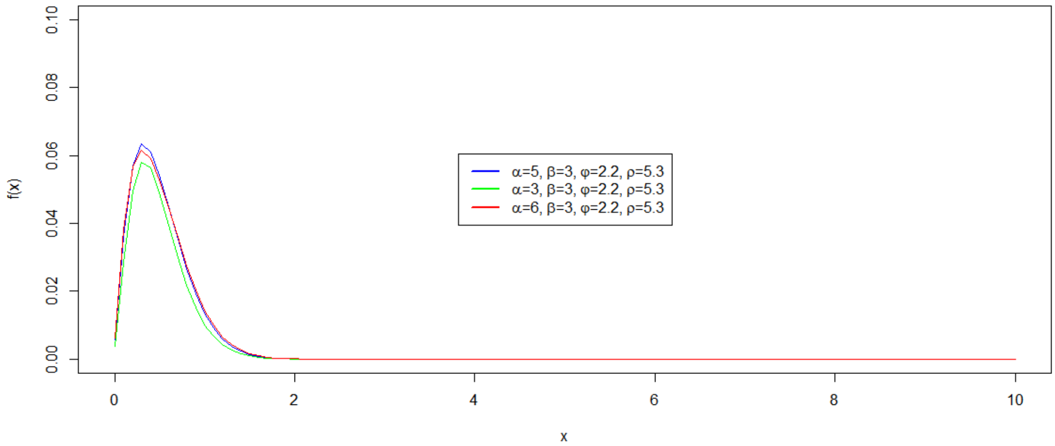


Fig. 2. Behaviour of pdf of KJ with varying alpha value

3.3 Linear representation of probability function of Kumaraswamy-Janardan distribution

Due to repetitive nature of pdf, determination of statistical properties becomes repetitive and time consuming. To reduce this complexity, the pdf is transformed as linear representation using binomial series expansion as demonstrated in this section.

Recall equation (7)

$$f(x) = \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\left(\frac{\rho x}{\varphi}-1\right)\right] \times \left[\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\,xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right]\left[\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\,xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right]e^{-\frac{\rho}{\varphi}x}$$

In mathematics, binomial series of $(1+b)^p\ ;\ p\ >\ 0, \in R,$ can be written linearly as $\sum_{i=0}^{n}\binom{p}{i}\,b^i$.

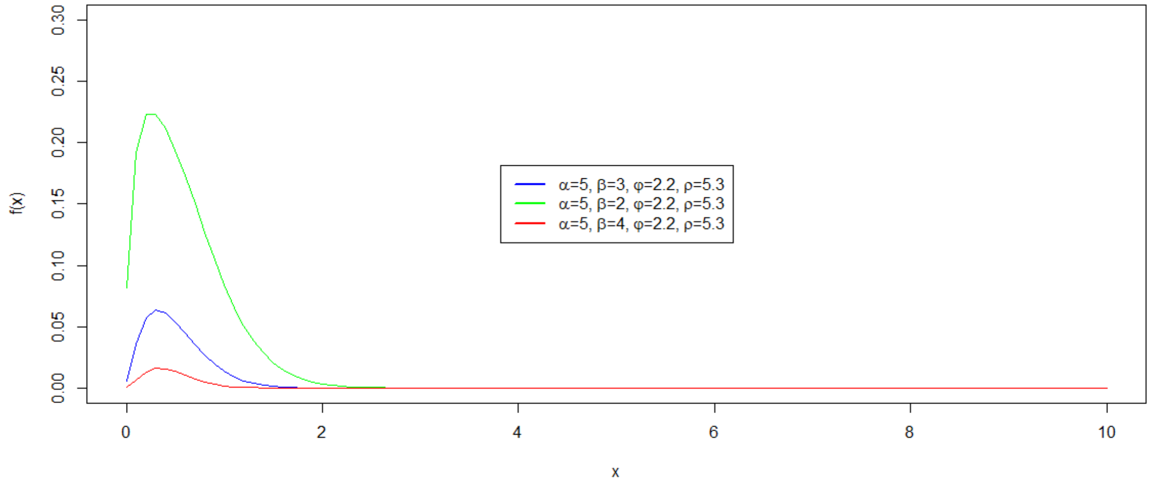


Fig. 3. Behaviour of pdf of KJ with varying beta value

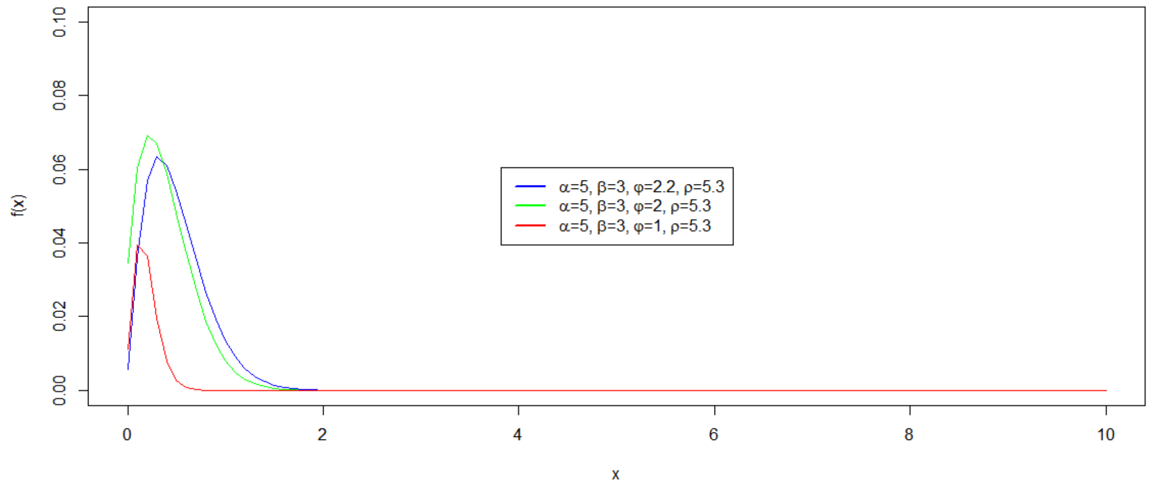


Fig. 4. Behaviour of pdf of KJ with varying varphi value

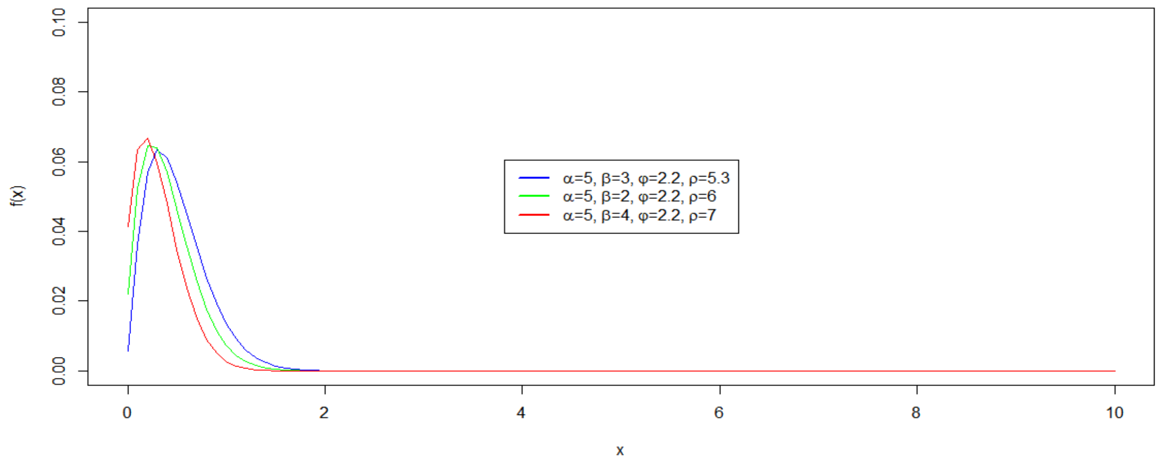


Fig. 5. Behaviour of pdf of KJ with varying rho value

Hence Binomial series presentation employed in simplification of the pdf of Kumaraswamy distribution is:

$$(1+b)^p = \sum_{i=0}^{n}\binom{p}{i} b^i \tag{10}$$

Applying equation (10) to (7),

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha i} \times \left[\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}$$

Simplifying to separate the variable x from the constants, we have:

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}\left(xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha i} \times \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left(e^{-\frac{\rho}{\varphi}x}\right)^{\alpha i} \times \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left(e^{-\alpha i\frac{\rho}{\ddot{o}}x}\right)\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-\frac{\rho}{\varphi}x}\left(e^{-\alpha i\frac{\rho}{\varphi}x}\right)$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-(1+\alpha i)\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right] e^{-(1+\alpha i)\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1}(x)^{\alpha-1}\right] e^{-(1+\alpha i)\frac{\rho}{\varphi}x}\, e^{-\frac{\rho}{\varphi}(\alpha-1)x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1}(x)^{\alpha-1}\right] e^{-((1+\alpha i)\frac{\rho}{\varphi}x+(\alpha-1)\frac{\rho}{\varphi}x)}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1}(x)^{\alpha-1}\right] e^{-((1+\alpha i)+(\alpha-1))\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1}(x)^{\alpha-1}\right] e^{-((\alpha i)+(\alpha))\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}(x)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1}(x)^{\alpha-1}\right] e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}$$

$$f(x) = \sum_{i=0}^{\beta-1}\binom{\beta-1}{i}(-1)^{\alpha i}\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i}\left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1}\right](x)^{\alpha i+\alpha-1} e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}$$

Let

$$A_i = \sum_{i=0}^{\beta-1} \binom{\beta-1}{i} (-1)^{\alpha i} \left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha i} \left[\left(\frac{-\rho\varphi^2}{\varphi(\rho+\varphi^2)}\right)^{\alpha-1}\right]$$

Then

$$f(x) = A_i \; x^{(\alpha i+\alpha-1)} \; e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x} \tag{11}$$

3.4 Statistical properties of Kumaraswamy-Janardan distribution

In this section, we introduce some statistical properties of Kumaraswamy-Janardan distribution. Some of the statistical properties established are moments about the origin and maximum likelihood estimates of the parameters.

3.4.1 Moment and moment generating function

For a random variable X that follows Kumaraswamy-Janardan distribution, its pdf is stated as:

$$f(x) = \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\left(\frac{\rho x}{\varphi}-1\right)\right] \times \left[\left(1-\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right]\left[\left(-\frac{\rho\varphi^2}{\varphi(\rho+\varphi^2)}\, xe^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1}\right]e^{-\frac{\rho}{\varphi}x}$$

With its linear representation as:

$$f(x) = A_i \; x^{(\alpha i+\alpha-1)} \; e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}$$

The r^th moment about the origin is defined as:

$$E(X^r) = \int_0^\infty x^{\,r} f(x)\, dx \tag{12}$$

Substituting (11) into (12), we obtain:

$$E(X^r) = A_i \int_0^\infty x^{\,r} \; x^{(\alpha i+\alpha-1)} \; e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}\, dx$$

$$E(X^r) = A_i \int_0^\infty x^{(\alpha i+\alpha+r)-1} \; e^{-(\alpha i+\alpha)\frac{\rho}{\varphi}x}\, dx \tag{13}$$

Using gamma transformation;

$$Gamma(\alpha,\beta) = \int_0^\infty x^{\alpha-1}e^{-\frac{1}{\beta}x} = \lceil\alpha \times \beta^\alpha = (\alpha-1)!\,\beta^\alpha$$

$$E(X^r) = A_i\left[(\alpha i+\alpha+r)!\,\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+r+1}\right] \tag{14}$$

$$E(X^r) = \begin{cases} A_i\left[(\alpha i+\alpha+r)!\,\left(\dfrac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+r+1}\right], & r = 1,2,3,\ldots \\ 0, & elsewhere \end{cases}$$

From (14), it follows, therefore, that:

$$E(X) = A_i\left[(\alpha i+\alpha+1)!\,\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+2}\right]$$

$$E(X^2) = A_i\left[(\alpha i+\alpha+2)!\,\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+3}\right]$$

$$E(X^3) = A_i\left[(\alpha i+\alpha+3)!\,\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+4}\right]$$
$$E(X^4) = A_i\left[(\alpha i+\alpha+4)!\,\left(\frac{\varphi}{(\alpha i+\alpha)\rho}\right)^{\alpha i+\alpha+5}\right]$$

Moment generating function of KJ distribution is given by

$$M_x(t) = \int_0^\infty e^{tx} f(x)\,dx$$

Using the fact that $e^{tx} = \sum_{r=0}^{\infty}\frac{(tx)^r}{r!}$, we get

$$M_x(t) = \int_0^\infty \sum_{r=0}^{\infty}\frac{(tx)^r}{r!} f(x)\,dx$$
$$M_x(t) = \int_0^\infty \sum_{r=0}^{\infty}\frac{(t)^r}{r!} x^r f(x)\,dx$$
$$M_x(t) = \sum_{r=0}^{\infty}\frac{(t)^r}{r!}\int_0^\infty x^r f(x)\,dx$$
$$M_x(t) = \sum_{r=0}^{\infty}\frac{(t)^r}{r!} E(X^r) \tag{15}$$

3.4.2 Maximum Likelihood Estimation (MLE)

In probability theory, Maximum Likelihood Estimation is used to estimate parameters of probability distributions given some observed sample data. According to Rossi [19], maximizing the likelihood function of probability model, the sample data is most probable. The estimated point in the parameter space that maximizes the likelihood function is called maximum likelihood estimate. Any given set of observations is a sample from unknown population, and MLE helps draw inferences about the population that most likely generated the sample [20].

Given the background discussions, this section presents the maximum likelihood estimation of Kumaraswamy-Janardan Distribution.

Intuitively, given parameter space $\emptyset = [\alpha\ \ \beta\ \ \rho\ \ \varphi]^T$, MLE $L(\emptyset, x)$ is given as:

$$L(\emptyset, x) = \prod_{i=1}^{n} f(x_i/\emptyset) \tag{16}$$

Plugging (7) into (16), we obtain

$$L(\emptyset,x) = \prod_1^n\left\{\left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right]\left[\frac{\rho x}{\varphi}-1\right]\times\left[\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right]\times\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1} e^{-\frac{\rho}{\varphi}x}\right\}$$

$$L(\emptyset,x) = \left[\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right]^n\prod_1^n\left\{\left[\frac{\rho x}{\varphi}-1\right]\times\left[\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right]\times\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha-1} e^{-\frac{\rho}{\varphi}x}\right\}$$

Taking "ln" on both side, we obtain log-likelihood ($Logl$) and it is given as:

$$l = n\left(\ln\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right)+\sum_{i=1}^{n} ln\left\{\left[\frac{\rho x}{\varphi}-1\right]\times\left[\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right)^{\beta-1}\right]\times\left(1-\right.\right.$$
$\varphi\rho+\varphi 2+\rho\varphi 2x\ \varphi\rho+\varphi 2\ e-\rho\varphi x\alpha-1e-\rho\varphi x$

$$l = n\left(\ln\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right)+\sum_{i=1}^{n}\left\{\begin{array}{c} ln\left[\frac{\rho x}{\varphi}-1\right]+(\beta-1)ln\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right) \\ +(\alpha-1)\ln\ (1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x})\ -\frac{\rho}{\varphi}x \end{array}\right\}$$

$$l = n\left(\ln\frac{\alpha\beta\rho\varphi}{(\rho+\varphi^2)}\right) + \sum_{i=1}^{n}\left\{\begin{matrix} ln\left[\frac{\rho x-\varphi}{\varphi}\right] + (\beta-1)ln\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right) \\ +(\alpha-1)\ln\ (1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x})\ -\frac{\rho}{\varphi}x \end{matrix}\right\}$$

$$l = n[\ln(\alpha) + \ln(\beta) + \ln(\rho) + \ln(\varphi) - \ln\,(\rho+\varphi^2)] + \sum_{i=1}^{n}\left\{\begin{matrix} \ln(\rho x-\varphi) - \ln(\varphi) \\ +(\beta-1)ln\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right) \\ +(\alpha-1)\ln\ (1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x})\ -\frac{\rho}{\varphi}x \end{matrix}\right\}$$

$$l = n[\ln(\alpha) + \ln(\beta) + \ln(\rho) + \ln(\varphi) - \ln(\rho+\varphi^2)] + \sum_{1}^{n}\ln(\rho x-\varphi) - n\ln(\varphi)$$
$$+(\beta-1)\sum_1^n \ln\left(1-\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right)^{\alpha}\right) + (\alpha-1)\sum_1^n \ln\left(1-\frac{\varphi(\rho+\varphi^2)+\rho\varphi^2 x}{\varphi(\rho+\varphi^2)}\ e^{-\frac{\rho}{\varphi}x}\right) - \frac{\rho n\bar{x}}{\varphi}$$

The objective is to find the values of the respective parameters in the model that maximize the likelihood function over the parameter space, $\emptyset$. This objective can be achieved if the log-likelihood function is partially differentiated over the parameter space $\frac{\partial l}{\partial \alpha}$, $\frac{\partial l}{\partial \beta}$,$\frac{\partial l}{\partial \rho}$,$\frac{\partial l}{\partial \varphi}$.

The estimated point in the parameter space $(\hat{\varphi}, \hat{\rho}, \hat{\alpha}, \hat{\beta})$ that maximizes the likelihood function of KJ distribution is achieved by simultaneously solving the optimality equations:

$$\frac{\partial l}{\partial \alpha} = 0,\ \frac{\partial l}{\partial \beta} = 0,\ \ \frac{\partial l}{\partial \rho} = 0\quad and\ \frac{\partial l}{\partial \varphi} = 0$$

The optimality equations are complex to solve analytically but can be solved empirically using an iterative technique like Simulated Annealing (SANN) (an algorithmic optimisation routine in R-Studio).

4 Application

In this section, we present empirical optimization results of the proposed Kumaraswamy-Janardan distribution using real data in demonstrating that KJ model provides significant improvement over the sub-models (Lindley and Janardan).

From the table, the AIC for KJ is smallest in comparison with the sub-models.

Table 1. Empirical optimization result of Kumaraswamy-Janardan Model

Datasets	Model	$\hat{\rho}$	$\hat{\varphi}$	$\hat{\alpha}$	$\hat{\beta}$	-(log L)	AIC
Dataset 1	Lindley	2.9097	-	-	-	73.1047	75.1047
	Janardan	64.3228	4.6215	-	-	-125.7309	-121.7309
	K-J	4.3964	0.1174	13.0521	0.9394	-231.1150	-223.1150
Dataset 2	Lindley	2.9097	-	-	-	-28.8451	-26.8451
	Janardan	76.4254	4.9416	-	-	-33.5370	-29.5370
	K-J	0.1908	0.2571	7.0334	15.7885	-39.7005	-31.7005
Dataset 3	Lindley	2.7243	-	-	-	95.9087	97.9087
	Janardan	71.0587	4.8012	-	-	37.0897	78.1794
	K-J	2.7607	10.8387	5.5524	9.6799	29.5206	67.0522

5 Conclusions

In this paper, we define a new four-parameter probability distribution known as Kumaraswamy-Janardan (KJ) Distribution of which Janardan distribution is a particular case. The probability density function, cumulative density function, survivor rate function as well as hazard rate function of the distribution are established. Statistical properties such as moments, moment generating function as well as maximum likelihood of the model are discussed. The model is tested with real data and is proven to be more flexible in fitting real data than any of its sub-models.

Competing Interests

Authors have declared that no competing interests exist.

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