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Cyclic Solution and Optimal Approximation of the Quaternion Stein Equation

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Abstract

In this paper, two different methods are used to study the cyclic structure solution and the optimal approximation of the quaternion Stein equation $AXB - X = F$. Firstly, the matrix equation equivalent to the target structure matrix is constructed by using the complex decomposition of the quaternion matrix, to obtain the necessary and sufficient conditions for the existence of the cyclic solution of the equation and the expression of the general solution. Secondly, the Stein equation is converted into the Sylvester equation by adding the necessary parameters, and the condition for the existence of a cyclic solution and the expression of the equation's solution are then obtained by using the real decomposition of the quaternion matrix and the Kronecker product of the matrix. At the same time, under the condition that the solution set is non-empty, the optimal approximation solution to the given quaternion circulant matrix is obtained by using the property of Frobenius norm property. Numerical examples are given to verify the correctness of the theoretical results and the feasibility of the proposed method.

Keywords

Quaternion Field, Stein Equation, Cyclic Matrix, Complex Decomposition, Real Decomposition, Optimal Approximation

1. Introduction

In the field of numerical algebra, it is frequently important to discuss some structural solutions of a matrix equation. For example, [1] studied the symmetric solution of the equation Lyapunov $AX + YA = C$ in the real field; [2] gave the η-inverse Hermitian solution of a class of classical matrix equations in the quaternion field; and [3] [4] [5] studied the cyclic solution and unitary structure solution of the Lyapunov equation and the Sylvester equation on the quaternion field.

In the domains of cybernetics, system stability analysis, probability statistics, spectral analysis, neural networks, and image restoration, the Stein equation is a type of matrix equation that is frequently utilized [6] [7] [8] [9] [10]. Numerous academics have investigated the equation's general solution as well as a few structural solutions using various techniques, with some success. For example, the positive definite solution of the Stein equation was discussed in 2012 [11], the general solution of the Stein equation was given by using the double conjugate gradient method in 2019 [12], and the cyclic solution of the Stein equation was discussed in the complex field by using the H-representation method of the matrix in 2022 [13]. However, there is no related research report on the cyclic solution of the Stein equation over the quaternion field, so the study of the cyclic structure solution of the quaternion Stein equation is a novel topic. The purpose of this paper is to discuss the cyclic solution of Stein equation $\boldsymbol{AXB}-\boldsymbol{X}=\boldsymbol{F}$ and its optimal approximation solution on the quaternion field.

Let $\mathbf{Q}^{n\times n},\mathbf{C}^{n\times n},\mathbf{R}^{n\times n}$ denote the set of all $n\times n$ quaternion matrices, complex matrices and real matrices; let $\overline{\boldsymbol{A}},\boldsymbol{A}^T,\boldsymbol{A}^*$ denote the conjugacy, transpose and conjugate transpose of matrix $\boldsymbol{A}$, and $\boldsymbol{A}^+$ denotes the Moore-Penrose generalized inverse of matrix $\boldsymbol{A}$; let $\otimes$ denote the Kronecker product and $vec(\boldsymbol{A})$ denote the vector that straightens the columns of matrix $\boldsymbol{A}$ sequentially. And let $\|\boldsymbol{A}\|=[\mathrm{tr}(\boldsymbol{A}^H\boldsymbol{A})]^{1/2}=[\sum_{i,j=1}^{n}|a_{ij}|^2]^{1/2}$ denote the Frobenius norm of quaternion matrix $\boldsymbol{A}$ [14] and $x_0=x(1:n)$ denote an n-dimensional vector consisting of the first to nth elements of vector *x*. This paper mainly discusses the following two issues.

Problem 1. Given the matrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{F}\in\mathbf{Q}^{n\times n}$, finding the circulant matrix $\boldsymbol{X}\in\mathbf{Q}^{n\times n}$ makes

$$\boldsymbol{AXB}-\boldsymbol{X}=\boldsymbol{F} \tag{1}$$

Problem 2. Given the circulant matrix $\boldsymbol{P}\in\mathbf{Q}^{n\times n}$, the quaternion circulant matrix $\tilde{\boldsymbol{X}}\in\Omega$ is obtained under the condition of the solution set of problem 1 and the $\Omega\neq\varnothing$, such that

$$\left\|\tilde{\boldsymbol{X}}-\boldsymbol{P}\right\|=\min_{\boldsymbol{X}\in\Omega}\left\|\boldsymbol{X}-\boldsymbol{P}\right\|. \tag{2}$$

2. Related Definitions and Lemmas

Define 1. Given quaternion matrix $\boldsymbol{A}=\boldsymbol{A}_1+\boldsymbol{A}_2 j\in\mathbf{Q}^{m\times n}(\boldsymbol{A}_1,\boldsymbol{A}_2\in\mathbf{C}^{m\times n})$, the form

$$L(\boldsymbol{A})=\begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ -\overline{\boldsymbol{A}}_2 & \overline{\boldsymbol{A}}_1\end{bmatrix}\in\mathbf{C}^{2m\times 2n} \tag{3}$$

called to be the complex representation matrix of matrix $\boldsymbol{A}$.

Define 2 An $n\times n$ matrix

$$\boldsymbol{X}=\begin{bmatrix}q_0 & q_1 & \cdots & q_{n-1}\\ q_{n-1} & q_0 & \cdots & q_{n-2}\\ \vdots & \vdots & \ddots & \vdots\\ q_1 & q_2 & \cdots & q_0\end{bmatrix}\in\mathbf{Q}^{n\times n} \tag{4}$$

is called n-order quaternion circulant matrix. Then $X = Cir(q_0, q_1, \cdots, q_{n-1})$, note

$$D = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix} \in \mathbf{R}^{n\times n} \tag{5}$$

where I_{n-1} is a unit matrix, Then the circulant matrix (4) can be expressed as

$$X = q_{n-1}D^{n-1} + \cdots + q_1 D + q_0 I = (D^{n-1}q, D^{n-2}q, \cdots, q) \tag{6}$$

where

$$q = (q_{n-1}, \cdots, q_1, q_0)^{\mathrm{T}} \in \mathbf{Q}^{n\times 1}. \tag{7}$$

Lemma 1. [15] For any matrix $A, B \in \mathbf{Q}^{n\times n}$, according to Definition 1, it is easy to prove that the complex representation matrix of quaternion matrix has the following properties.

(a) $A = B \Leftrightarrow L(A) = L(B)$; $L(A+B) = L(A) + L(B)$; $L(AB) = L(A)L(B)$;

(b) $\overline{L(A)} = Q_{2n}^{-1} \cdot L(A) \cdot Q_{2n}$, where $Q_{2n} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$.

Lemma 2. [16] The matrix equation $AX = B$ over the complex field has a solution if and only if $AA^{+}B = B$. When there is a solution, both the general solution and the least square solution of the equation can be expressed as $X = A^{+}B + (I - A^{+}A)Q$, where Q is an arbitrary matrix and $\tilde{X} = A^{+}B$ is the unique minimum norm least square solution.

Lemma 3. Let the quaternion matrix $A, B, F \in \mathbf{Q}^{n\times n}$, then the quaternion Stein equation $AXB - X = F$ has a solution if and only if its complex representation equation

$$L(A)YL(B) - Y = L(F) \tag{8}$$

has a solution. Where $Y \in \mathbf{C}^{2n\times 2n}$, If Y is the solution of Equation (8), then

$$X = \frac{1}{4}[I_n, -jI_n] \cdot \left[Y + Q_{2n}\overline{Y}Q_{2n}^{-1}\right] \cdot \begin{bmatrix} I_n \\ jI_n \end{bmatrix} \tag{9}$$

is the solution of the original equation $AXB - X = F$, where Q_{2n} such as Lemma 1(b).

Proof. It is obvious to prove the necessity, when $X \in \mathbf{Q}^{n\times n}$ is the solution of equation (1), $Y = L(X) \in \mathbf{C}^{2n\times 2n}$ must be the solution of Equation (8).

For the adequacy, let $Y \in \mathbf{C}^{2n\times 2n}$ be the solution of Equation (8), and divide Y into the following blocks:

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, Y_i \in \mathbf{C}^{n\times n}, (i = 1, 2, 3, 4) \tag{10}$$

It is sufficient to prove that the matrix (9) determined by Equation (10) is the solution of the original equation. The formula (10) is substituted into the (9), and it is calculated.

$$X = \frac{1}{2}\left(Y_1 + \overline{Y}_4\right) + \frac{1}{2}\left(Y_2 - \overline{Y}_3\right)j.$$

From Lemma 1,

$$L(\boldsymbol{X})=\frac{1}{2}\begin{bmatrix}\boldsymbol{Y}_1+\overline{\boldsymbol{Y}}_4 & \boldsymbol{Y}_2-\overline{\boldsymbol{Y}}_3\\ -\overline{\boldsymbol{Y}}_2+\boldsymbol{Y}_3 & \overline{\boldsymbol{Y}}_1+\boldsymbol{Y}_4\end{bmatrix}=\frac{1}{2}\left(\boldsymbol{Y}+\boldsymbol{Q}_{2n}\overline{\boldsymbol{Y}}\boldsymbol{Q}_{2n}^{-1}\right)$$

and $\boldsymbol{Y}$ satisfies the Equation (8), after calculation

$$L(\boldsymbol{A})L(\boldsymbol{X})L(\boldsymbol{B})-L(\boldsymbol{X})=\frac{1}{2}L(\boldsymbol{F})+\frac{1}{2}\left(\boldsymbol{Q}_{2n}\overline{L(\boldsymbol{F})}\boldsymbol{Q}_{2n}^{-1}\right)=L(\boldsymbol{F}).$$

Therefore, (9) determined by Equation (10) is the solution of the original equation.

3. The Solution of Problem 1

In this section, we discuss the necessary and sufficient conditions for the existence of cyclic solutions of quaternion matrix Equation (1) and the expressions of their general solutions. Here are two ways to discuss it.

3.1. Complex Representation

Define

$$\boldsymbol{M}=\begin{bmatrix}\boldsymbol{D}^{n-1} & 0\\ 0 & \boldsymbol{D}^{n-1}\\ \boldsymbol{D}^{n-2} & 0\\ 0 & \boldsymbol{D}^{n-2}\\ \vdots & \vdots\\ \boldsymbol{I}_n & 0\\ 0 & \boldsymbol{I}_n\end{bmatrix}\in\mathbf{R}^{2n^2\times 2n},\tilde{\boldsymbol{D}}=\begin{bmatrix}\boldsymbol{M} & 0\\ 0 & \boldsymbol{M}\end{bmatrix},\boldsymbol{W}=\left((L(\boldsymbol{B}))^T\otimes L(\boldsymbol{A})-\boldsymbol{I}_{4n^2}\right). \quad (11)$$

First of all, the complex representation equation $L(\boldsymbol{A})\boldsymbol{Y}L(\boldsymbol{B})-\boldsymbol{Y}=L(\boldsymbol{F})$ of matrix Equation (1) is equivalently expressed to using Kronecker product as

$$\left((L(\boldsymbol{B}))^T\otimes L(\boldsymbol{A})-\boldsymbol{I}_{4n^2}\right)vec(\boldsymbol{Y})=vec(L(\boldsymbol{F})). \quad (12)$$

Using the expression of circulant matrix (6) and n-order complex circulant matrix $\boldsymbol{Y}_1,\boldsymbol{Y}_2,\boldsymbol{Y}_3,\boldsymbol{Y}_4$ in (10), then

$$\boldsymbol{Y}_i=\left(\boldsymbol{D}^{n-1}y_i,\boldsymbol{D}^{n-2}y_i,\cdots,y_i\right)\quad(i=1,2,3,4) \quad (13)$$

where $y_i\in\mathbf{C}^{n\times 1}$ is the last column elements of $\boldsymbol{Y}_i\,(i=1,2,3,4)$. Therefore, from the definitions of (11) and (13) and of the straightening operation we have

$$vec(\boldsymbol{Y})=\tilde{\boldsymbol{D}}y,\quad y=(y_1,y_3,y_2,y_1)^T. \quad (14)$$

In summary, the matrix Equation (12) is equivalent to

$$\boldsymbol{W}\tilde{\boldsymbol{D}}y=vec\left(L(\boldsymbol{F})\right) \quad (15)$$

So with regard to the solution of problem 1, we have the following results.

Theorem 1. Given the matrix $\boldsymbol{A},\boldsymbol{B},\boldsymbol{F}\in\mathbf{Q}^{n\times n}$, and the Stein equation $\boldsymbol{AXB}-\boldsymbol{X}=\boldsymbol{F}$ has a cyclic solution if and only if

$$\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}vec\left(L(\boldsymbol{F})\right)=vec\left(L(\boldsymbol{F})\right). \quad (16)$$

When there is a solution

$$\boldsymbol{X}=\frac{1}{4}\left[\boldsymbol{I}_n,-j\boldsymbol{I}_n\right]\cdot\left[\boldsymbol{Y}+\boldsymbol{Q}_{2n}\bar{\boldsymbol{Y}}\boldsymbol{Q}_{2n}^{-1}\right]\cdot\begin{bmatrix}\boldsymbol{I}_n\\ j\boldsymbol{I}_n\end{bmatrix} \tag{17}$$

where

$$y=\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}vec\left(L\left(\boldsymbol{F}\right)\right)+\left[\boldsymbol{I}_{4n}-\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\right]z,\forall z\in\mathbf{C}^{4n\times 1}$$
$$y_1=y(1:n),y_3=y(n+1:2n),y_2=y(2n+1:3n),y_4=y(3n+1:4n).$$

Proof From (11) - (15) and Lemma 2, the Equation (1) has a cyclic solution $\Leftrightarrow$ the complex matrix Equation (15) has solution $\Leftrightarrow$ $\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}vec\left(L\left(\boldsymbol{F}\right)\right)=vec\left(L\left(\boldsymbol{F}\right)\right)$. Then it is known from Lemma 2 and Lemma 3 that when the cyclic solution of the matrix equation $\boldsymbol{AXB}-\boldsymbol{X}=\boldsymbol{F}$ exists, its general expression is shown in (17).

3.2. Parameter Transformation Method

The real parameter α is chosen such that $\boldsymbol{B}+\alpha\boldsymbol{I}$ and $\alpha\boldsymbol{A}+\boldsymbol{I}$ are simultaneously invertible, so the Equation (1) can be equivalently deformed into

$$\begin{gathered}\boldsymbol{AX}(\boldsymbol{B}+\alpha\boldsymbol{I})-(\alpha\boldsymbol{A}+\boldsymbol{I})\boldsymbol{X}=\boldsymbol{F}\\ \Rightarrow(\alpha\boldsymbol{A}+\boldsymbol{I})^{-1}\boldsymbol{AX}-\boldsymbol{X}(\boldsymbol{B}+\alpha\boldsymbol{I})^{-1}=(\alpha\boldsymbol{A}+\boldsymbol{I})^{-1}\boldsymbol{F}(\boldsymbol{B}+\alpha\boldsymbol{I})^{-1}\end{gathered} \tag{18}$$

Write down $\tilde{\boldsymbol{A}}=(\alpha\boldsymbol{A}+\boldsymbol{I})^{-1}\boldsymbol{A}$, $\tilde{\boldsymbol{B}}=(\boldsymbol{B}+\alpha\boldsymbol{I})^{-1}$, $\tilde{\boldsymbol{F}}=(\alpha\boldsymbol{A}+\boldsymbol{I})^{-1}\boldsymbol{F}(\boldsymbol{B}+\alpha\boldsymbol{I})^{-1}$, then (18) becomes

$$\tilde{\boldsymbol{A}}\boldsymbol{X}-\boldsymbol{X}\tilde{\boldsymbol{B}}=\tilde{\boldsymbol{F}} \tag{19}$$

It is obvious that (19) is a Sylvester equation. Let the real decomposition of the circulant matrix $\boldsymbol{X}\in\mathbf{Q}^{n\times n}$ be $\boldsymbol{X}=\boldsymbol{X}_0+\boldsymbol{X}_1i+\boldsymbol{X}_2j+\boldsymbol{X}_3k$ and the $\boldsymbol{X}_i\in\mathbf{R}^{n\times n}$ $(i=0,1,2,3)$ is real circulant matrix. Let the real decomposition of quaternion matrix $\tilde{\boldsymbol{A}},\tilde{\boldsymbol{B}},\tilde{\boldsymbol{F}}$ is

$$\tilde{\boldsymbol{A}}=\boldsymbol{A}_0+\boldsymbol{A}_1i+\boldsymbol{A}_2j+\boldsymbol{A}_3k,\ \tilde{\boldsymbol{B}}=\boldsymbol{B}_0+\boldsymbol{B}_1i+\boldsymbol{B}_2j+\boldsymbol{B}_3k,\ \tilde{\boldsymbol{F}}=\boldsymbol{F}_0+\boldsymbol{F}_1i+\boldsymbol{F}_2j+\boldsymbol{F}_3k$$

where $\boldsymbol{A}_i,\boldsymbol{B}_i,\boldsymbol{F}_i\in\mathbf{R}^{n\times n}\ (i=0,1,2,3)$, so the matrix Equation (19) is equivalent to

$$\begin{aligned}&\left(\boldsymbol{A}_0+\boldsymbol{A}_1i+\boldsymbol{A}_2j+\boldsymbol{A}_3k\right)\left(\boldsymbol{X}_0+\boldsymbol{X}_1i+\boldsymbol{X}_2j+\boldsymbol{X}_3k\right)\\ &-\left(\boldsymbol{X}_0+\boldsymbol{X}_1i+\boldsymbol{X}_2j+\boldsymbol{X}_3k\right)\left(\boldsymbol{B}_0+\boldsymbol{B}_1i+\boldsymbol{B}_2j+\boldsymbol{B}_3k\right)=\boldsymbol{F}_0+\boldsymbol{F}_1i+\boldsymbol{F}_2j+\boldsymbol{F}_3k\end{aligned} \tag{20}$$

expanding Equation (20) by using the uniqueness of quaternion real decomposition

$$\begin{cases}\boldsymbol{A}_0\boldsymbol{X}_0-\boldsymbol{X}_0\boldsymbol{B}_0-\boldsymbol{A}_1\boldsymbol{X}_1+\boldsymbol{X}_1\boldsymbol{B}_1-\boldsymbol{A}_2\boldsymbol{X}_2+\boldsymbol{X}_2\boldsymbol{B}_2-\boldsymbol{A}_3\boldsymbol{X}_3+\boldsymbol{X}_3\boldsymbol{B}_3=\boldsymbol{F}_0\\ \boldsymbol{A}_1\boldsymbol{X}_0-\boldsymbol{X}_0\boldsymbol{B}_1+\boldsymbol{A}_0\boldsymbol{X}_1-\boldsymbol{X}_1\boldsymbol{B}_0-\boldsymbol{A}_3\boldsymbol{X}_2-\boldsymbol{X}_2\boldsymbol{B}_3+\boldsymbol{A}_2\boldsymbol{X}_3+\boldsymbol{X}_3\boldsymbol{B}_2=\boldsymbol{F}_1\\ \boldsymbol{A}_2\boldsymbol{X}_0-\boldsymbol{X}_0\boldsymbol{B}_2+\boldsymbol{A}_3\boldsymbol{X}_1+\boldsymbol{X}_1\boldsymbol{B}_3+\boldsymbol{A}_0\boldsymbol{X}_2-\boldsymbol{X}_2\boldsymbol{B}_0-\boldsymbol{A}_1\boldsymbol{X}_3-\boldsymbol{X}_3\boldsymbol{B}_1=\boldsymbol{F}_2\\ \boldsymbol{A}_3\boldsymbol{X}_0-\boldsymbol{X}_0\boldsymbol{B}_3-\boldsymbol{A}_2\boldsymbol{X}_1-\boldsymbol{X}_1\boldsymbol{B}_2+\boldsymbol{A}_1\boldsymbol{X}_2+\boldsymbol{X}_2\boldsymbol{B}_1+\boldsymbol{A}_0\boldsymbol{X}_3-\boldsymbol{X}_3\boldsymbol{B}_0=\boldsymbol{F}_3\end{cases} \tag{21}$$

Because $\boldsymbol{X}_i\in\mathbf{R}^{n\times n}\ (i=0,1,2,3)$ is the real circulant matrix, according to (6), it is possible to order

$$\boldsymbol{X}_i=\left(\boldsymbol{D}^{n-1}x_i,\boldsymbol{D}^{n-2}x_i,\cdots,x_i\right),(i=0,1,2,3) \tag{22}$$

where $x_i\in\mathbf{R}^{n\times 1}$ is the last column elements of $\boldsymbol{X}_i$. Put

$$K=\begin{bmatrix} I\otimes A_0-B_0^T\otimes I & -I\otimes A_1+B_1^T\otimes I & -I\otimes A_2+B_2^T\otimes I & -I\otimes A_3+B_3^T\otimes I \\ I\otimes A_1-B_1^T\otimes I & I\otimes A_0-B_0^T\otimes I & -I\otimes A_3-B_3^T\otimes I & I\otimes A_2+B_2^T\otimes I \\ I\otimes A_2-B_2^T\otimes I & I\otimes A_3+B_3^T\otimes I & I\otimes A_0-B_0^T\otimes I & -I\otimes A_1-B_1^T\otimes I \\ I\otimes A_3-B_3^T\otimes I & -I\otimes A_2-B_2^T\otimes I & I\otimes A_1+B_1^T\otimes I & I\otimes A_0-B_0^T\otimes I \end{bmatrix}\mathbf{R}^{4n^2\times 4n^2}$$

$$\hat{D}=\begin{bmatrix} D^{n-1} \\ D^{n-2} \\ \vdots \\ I \end{bmatrix}\in\mathbf{R}^{n^2\times n},\tilde{K}=K\begin{bmatrix} \hat{D} & & & \\ & \hat{D} & & \\ & & \hat{D} & \\ & & & \hat{D} \end{bmatrix}\in\mathbf{R}^{4n^2\times 4n}, \tag{23}$$

$$L=[vec(F_0),vec(F_1),vec(F_2),vec(F_3)]^T\in\mathbf{R}^{4n^2\times 1},x=[x_0,x_1,x_2,x_3]^T\in\mathbf{R}^{4n\times 1}$$

Using the Kronecker product of a matrix, the system of Equation (21) is equivalent to

$$K\begin{bmatrix} \hat{D}x_0 \\ \hat{D}x_1 \\ \hat{D}x_2 \\ \hat{D}x_3 \end{bmatrix}=L \Leftrightarrow \tilde{K}x=L \tag{24}$$

where $\tilde{K}$ is represented by (23). So with regard to the solution of problem 1, we have the following results.

Theorem 2. Given the quaternion matrix $A,B,F\in\mathbf{Q}^{n\times n}$, the Stein equation $AXB-X=F$ has a cyclic solution if and only if $\tilde{K}\tilde{K}^{+}L=L$. When there is a solution, its general solution is

$$X=X_0+X_1i+X_2j+X_3k \tag{25}$$

where

$$x=\tilde{K}^{+}L+\left(I-\tilde{K}^{+}\tilde{K}\right)V,\ \ V\in\mathbf{R}^{4n\times 1}\ \text{ is arbitrary}$$

$$x_0=x(1:n),x_1=x(n+1:2n),x_3=x(2n+1:3n),x_0=x(3n+1:4n)$$

$$X_i=\left(D^{n-1}x_i,D^{n-2}x_i,\cdots,x_i\right)=C\left(x_i(n),x_i(n-1),\cdots,x_i(1)\right),(i=0,1,2,3)$$

Proof. From (22) - (24) and Lemma 2, the Equation (1) has a cyclic solution $\Leftrightarrow$ Sylvester Equation (19) has solution $\Leftrightarrow$ Equation (24) has solution $\Leftrightarrow \tilde{K}\tilde{K}^{+}L=L$, then it is known from Lemma 2 and Lemma 3 that when the cyclic solution of the Stein equation $AXB-X=F$ exists, its general solution expression is shown in (25).

4. The Solution of Problem 2

4.1. Complex Representation

Let $P\in\mathbf{Q}^{n\times n}$ be a known quaternion circulant matrix, may be set up

$$L(P)=R=\begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}.$$

Derived from (14)

$$vec\left(L(\boldsymbol{P})\right)=\tilde{\boldsymbol{D}}r,\quad r=[r_1,r_2,r_3,r_4]^T,$$

$$r_1=r(1:n),r_3=r(n+1:2n),r_2=r(2n+1:3n),r_4=r(3n+1:4n),$$

where $r_i\in\mathbf{C}^{n\times 1}$ is the last column elements of $\boldsymbol{R}_i\,(i=1,2,3,4)$. Write down

$$\boldsymbol{T}=\begin{bmatrix}0&0&0&\boldsymbol{I}_n\\0&0&-\boldsymbol{I}_n&0\\0&-\boldsymbol{I}_n&0&0\\\boldsymbol{I}_n&0&0&0\end{bmatrix}\in\mathbf{R}^{4n\times 4n}\tag{26}$$

Theorem 3. Let the $\Omega\neq\varnothing$ in question 1, and $\boldsymbol{P}\in\mathbf{Q}^{n\times n}$ is a known circulant matrix, then the solution $\tilde{\boldsymbol{X}}$ exists in Ω such that

$$\|\boldsymbol{X}-\boldsymbol{P}\|=\min$$

holds and has the following expression

$$\tilde{\boldsymbol{X}}=\frac{1}{2}\left(\boldsymbol{Y}_1+\overline{\boldsymbol{Y}}_4\right)+\frac{1}{2}\left(\boldsymbol{Y}_2-\overline{\boldsymbol{Y}}_3\right)j\tag{27}$$

where

$$y=\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}vec\left(L(\boldsymbol{F})\right)+\left[\boldsymbol{I}_{4n}-\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\right]z_0$$

$$y_1=y(1:n),y_3=y(n+1:2n),y_2=y(2n+1:3n),y_4=y(3n+1:4n)$$

$$\boldsymbol{Y}_i=\left(\boldsymbol{D}^{n-1}y_i,\boldsymbol{D}^{n-2}y_i,\cdots,y_i\right)(i=1,2,3,4).$$

Proof . When $\Omega\neq\varnothing$, according to Theorem 1, the cyclic solution of the equation $\boldsymbol{AXB}-\boldsymbol{X}=\boldsymbol{F}$ is shown in (17), and there is

$$L(\boldsymbol{X})=\frac{1}{2}\left(\boldsymbol{Y}+\boldsymbol{Q}_{2n}\overline{\boldsymbol{Y}}\boldsymbol{Q}_{2n}^{-1}\right)=\frac{1}{2}\begin{bmatrix}\boldsymbol{Y}_1+\overline{\boldsymbol{Y}}_4&\boldsymbol{Y}_2-\overline{\boldsymbol{Y}}_3\\-\overline{\boldsymbol{Y}}_2+\boldsymbol{Y}_3&\overline{\boldsymbol{Y}}_1+\boldsymbol{Y}_4\end{bmatrix}$$

according to (14) and (26)

$$vec(\boldsymbol{Y})=\tilde{\boldsymbol{D}}y,vec\left(\boldsymbol{Q}_{2n}\overline{\boldsymbol{Y}}\boldsymbol{Q}_{2n}^{-1}\right)=vec\begin{bmatrix}\overline{\boldsymbol{Y}}_4&-\overline{\boldsymbol{Y}}_3\\-\overline{\boldsymbol{Y}}_2&\overline{\boldsymbol{Y}}_1\end{bmatrix}=\tilde{\boldsymbol{D}}\boldsymbol{T}\overline{y}$$

and

$$\|\boldsymbol{X}-\boldsymbol{P}\|^2=\frac{1}{2}\left\|vec\left(L(\boldsymbol{X})\right)-vec\left(L(\boldsymbol{P})\right)\right\|^2=\frac{1}{2}\left\|\frac{1}{2}\tilde{\boldsymbol{D}}y+\frac{1}{2}\tilde{\boldsymbol{D}}\boldsymbol{T}\overline{y}-vec\left(L(\boldsymbol{P})\right)\right\|^2$$

$$=\frac{1}{8}\left\|\tilde{\boldsymbol{D}}y+\tilde{\boldsymbol{D}}\boldsymbol{T}\overline{y}-2vec\left(L(\boldsymbol{P})\right)\right\|^2=\frac{1}{8}\left\|\tilde{\boldsymbol{D}}\left(y+\boldsymbol{T}\overline{y}\right)-2\tilde{\boldsymbol{D}}r\right\|^2$$

then

$$\left\|\tilde{\boldsymbol{X}}-\boldsymbol{P}\right\|=\min\Leftrightarrow\left\|\tilde{\boldsymbol{D}}\left(y+\boldsymbol{T}\overline{y}\right)-2\tilde{\boldsymbol{D}}r\right\|^2=\min\tag{28}$$

that is when there is a unique minimum norm least square solution to the equation $\tilde{\boldsymbol{D}}y+\tilde{\boldsymbol{D}}\boldsymbol{T}\overline{y}=2\tilde{\boldsymbol{D}}r$, (28) is established.

$$y=\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}vec\left(L(\boldsymbol{F})\right)+\left[\boldsymbol{I}_{4n}-\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\right]z\tag{29}$$

get after substitution

$$\begin{aligned}&\tilde{\boldsymbol{D}}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+} vec\left(L\left(\boldsymbol{F}\right)\right)+\tilde{\boldsymbol{D}}\left[\boldsymbol{I}_{4n}-\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\right]z\\&+\tilde{\boldsymbol{D}}\boldsymbol{T}\left\{\overline{\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+} vec\left(L\left(\boldsymbol{F}\right)\right)+\left[\boldsymbol{I}_{4n}-\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\right]z}\right\}=2\tilde{\boldsymbol{D}}r.\end{aligned} \tag{30}$$

We write

$$\boldsymbol{G}=\tilde{\boldsymbol{D}}\left[\boldsymbol{I}_{4n}-\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\right],\boldsymbol{H}=\tilde{\boldsymbol{D}}\boldsymbol{T}\left[\overline{\boldsymbol{I}_{4n}-\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)}\right],$$

$$s=2\tilde{\boldsymbol{D}}r-\tilde{\boldsymbol{D}}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+} vec\left(L\left(\boldsymbol{F}\right)\right)-\tilde{\boldsymbol{D}}\boldsymbol{T}\overline{\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+} vec\left(L\left(\boldsymbol{F}\right)\right)}$$

then (30) can be transformed into

$$\boldsymbol{G}z+\boldsymbol{H}\overline{z}=s,$$

according to the real decomposition of the complex matrix and Lemma 2, it is obtained that there is a minimum norm least square solution for $\boldsymbol{G}z+\boldsymbol{H}\overline{z}=s$,

$$z_0=z_1+z_2 i\in\mathbf{C}^{4n\times 1} \tag{31}$$

where $\begin{bmatrix}z_1\\z_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{G}_1+\boldsymbol{H}_1 & \boldsymbol{H}_2-\boldsymbol{G}_2\\\boldsymbol{G}_2+\boldsymbol{H}_2 & \boldsymbol{G}_1-\boldsymbol{H}_1\end{bmatrix}^{+}\begin{bmatrix}s_1\\s_2\end{bmatrix}$, $\boldsymbol{G}_i,\boldsymbol{H}_i,z_i,s_i\ (i=1,2)$ are the real decomposition of $\boldsymbol{G},\boldsymbol{H},z,s$ respectively, replace z_0 with (29), we have

$$y=\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+} vec\left(L\left(\boldsymbol{F}\right)\right)+\left[\boldsymbol{I}_{4n}-\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^{+}\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\right]z_0.$$

From (31), there is a unique cyclic solution (27) for the problem 2.

4.2. Parameter Transformation Method

When the real decomposition of the circulant matrix $\boldsymbol{P}\in\mathbf{Q}^{n\times n}$ is $\boldsymbol{P}=\boldsymbol{P}_0+\boldsymbol{P}_1 i+\boldsymbol{P}_2 j+\boldsymbol{P}_3 k$, where $\boldsymbol{P}_i\in\mathbf{R}^{n\times n}\ (i=0,1,2,3)$ are real circulant matrices, let $p_i\in\mathbf{R}^{n\times 1}$ be the last column element of $\boldsymbol{P}_i$, then

$$\boldsymbol{P}_i=\left(\boldsymbol{D}^{n-1}p_i,\boldsymbol{D}^{n-2}p_i,\cdots,p_i\right)\left(i=0,1,2,3\right)$$

put

$$p=vec\left(p_0,p_1,p_2,p_3\right)\in\mathbf{R}^{4n\times 1} \tag{32}$$

because $\boldsymbol{D}=\begin{bmatrix}0 & \boldsymbol{I}_{n-1}\\1 & 0\end{bmatrix}\in\mathbf{R}^{n\times n}$ is a unitary matrix, according to the unitary product invariance of Frobenius norm, we obtain

$$\left\|\boldsymbol{D}^s x_i-\boldsymbol{D}^s q_i\right\|^2=\left\|x_i-q_i\right\|^2. \tag{33}$$

When $\boldsymbol{X}\in\Omega$, from Theorem 2 and Equation ((31), (32))

$$\begin{aligned}\left\|\boldsymbol{X}-\boldsymbol{P}\right\|^2&=\left\|\boldsymbol{X}_0+\boldsymbol{X}_1 i+\boldsymbol{X}_2 j+\boldsymbol{X}_3 k-\left(\boldsymbol{P}_0+\boldsymbol{P}_1 i+\boldsymbol{P}_2 j+\boldsymbol{P}_3 k\right)\right\|^2\\&=\sum_{i=0}^{3}\left\|\boldsymbol{X}_i-\boldsymbol{P}_i\right\|^2=\sum_{i=0}^{3}\left\|\left(\boldsymbol{D}^{n-1}x_i,\boldsymbol{D}^{n-2}x_i,\cdots,x_i\right)-\left(\boldsymbol{D}^{n-1}p_i,\boldsymbol{D}^{n-2}p_i,\cdots,p_i\right)\right\|^2\\&=\sum_{i=0}^{3}\sum_{s=0}^{n-1}\left\|\boldsymbol{D}^s x_i-\boldsymbol{D}^s p_i\right\|^2=n\sum_{i=0}^{3}\left\|x_i-p_i\right\|_F^2=n\left\|x-p\right\|^2\\&=n\left\|\tilde{\boldsymbol{K}}^{+}\boldsymbol{L}+\left(\boldsymbol{I}-\tilde{\boldsymbol{K}}^{+}\boldsymbol{K}\right)\boldsymbol{V}-q\right\|^2.\end{aligned} \tag{34}$$

Therefore, with regard to the solution of problem 2, there are the following results.

Theorem 4 Let the $\Omega \neq \varnothing$ of problem 1, $\boldsymbol{P} \in \mathbf{Q}^{n\times n}$ is a known circulant matrix, then solution $\tilde{\boldsymbol{X}}$ exists in Ω such that

$$\|\boldsymbol{X}-\boldsymbol{P}\| = \min$$

holds and has the following expression

$$\tilde{\boldsymbol{X}} = \boldsymbol{X}_0 + \boldsymbol{X}_1 i + \boldsymbol{X}_2 j + \boldsymbol{X}_3 k \tag{35}$$

where

$$x = \begin{cases} \tilde{\boldsymbol{K}}^+\boldsymbol{L} + \left(\boldsymbol{I}-\tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}}\right)\left(\boldsymbol{I}-\tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}}\right)^+\left(q-\tilde{\boldsymbol{K}}^+\boldsymbol{L}\right), & \tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}} \neq \boldsymbol{I} \\ \tilde{\boldsymbol{K}}^+\boldsymbol{L}, \quad \tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}} = \boldsymbol{I} \end{cases}$$

$$x_0 = x(1:n), x_1 = x(n+1:2n), x_2 = x(2n+1:3n), x_3 = x(3n+1:4n)$$

$$\boldsymbol{X}_i = \left(\boldsymbol{D}^{n-1}x_i, \boldsymbol{D}^{n-2}x_i, \cdots, x_i\right) = C\left(x_i(n), x_i(n-1), \cdots, x_i(1)\right), (i=0,1,2,3)$$

Proof. From (34), we have

$$\|\boldsymbol{X}-\boldsymbol{P}\|^2 = \min \Leftrightarrow \left\|\tilde{\boldsymbol{K}}^+\boldsymbol{L} + \left(\boldsymbol{I}-\tilde{\boldsymbol{K}}^+\boldsymbol{K}\right)\boldsymbol{V} - q\right\|^2 = \min,$$

according to Lemma 2, it is known that when $\tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}} \neq \boldsymbol{I}$, the above least square solution of $\boldsymbol{V}$ is $\hat{\boldsymbol{V}} = \left(\boldsymbol{I}-\tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}}\right)^+\left(x-\tilde{\boldsymbol{K}}^+\boldsymbol{L}\right)$, which can be obtained from Theorem 2,

$$x = \begin{cases} \tilde{\boldsymbol{K}}^+\boldsymbol{L} + \left(\boldsymbol{I}-\tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}}\right)\left(\boldsymbol{I}-\tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}}\right)^+\left(q-\tilde{\boldsymbol{K}}^+\boldsymbol{L}\right), & \tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}} \neq \boldsymbol{I} \\ \tilde{\boldsymbol{K}}^+\boldsymbol{L}, \quad \tilde{\boldsymbol{K}}^+\tilde{\boldsymbol{K}} = \boldsymbol{I} \end{cases}$$

Therefore, the exist $\tilde{\boldsymbol{X}} \in \Omega$ such that $\|\boldsymbol{X}-\boldsymbol{P}\|_F = \min$ holds and the expression for $\tilde{\boldsymbol{X}}$ is shown in (35).

5. Solving Steps

According to the results of Theorem 1 and Theorem 2, we give the following steps for solving problem 1 and problem 2 (taking the complex representation as an example).

- For a given quaternion matrix $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{F} \in \mathbf{Q}^{n\times n}$, write their complex representation matrix, that is $L(\boldsymbol{A}), L(\boldsymbol{B}), L(\boldsymbol{F}) \in \mathbf{C}^{2n\times 2n}$.
- Write out $\boldsymbol{M}, \tilde{\boldsymbol{D}}, \boldsymbol{W}$ according to (11).
- Test whether condition $\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)\left(\boldsymbol{W}\tilde{\boldsymbol{D}}\right)^+ vec\left(L(\boldsymbol{F})\right) = vec\left(L(\boldsymbol{F})\right)$ holds. If true, problem 1 has a solution, otherwise problem 1 has no solution.
- According to the result of Theorem 1, the cyclic matrix (17) is written, that is, the cyclic solution $\boldsymbol{X}$ of problem 1 is obtained.
- When problem 1 has a solution, write the complex representation matrix of the circulant matrix $\boldsymbol{P}$, write the vector y according to formula (14), and then write the best approximate solution $\tilde{\boldsymbol{X}}$ to $\boldsymbol{P}$ in Ω according to formula (27).

6. Numerical Example

Example. Given the matrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{F}\in\mathbf{Q}^{3\times3}$ are as follows

$$\boldsymbol{A}=\begin{bmatrix}1 & i & j\\ i & 2 & k\\ 0 & i & 3\end{bmatrix},\boldsymbol{B}=\begin{bmatrix}1 & 0 & i\\ 2 & i & j\\ 0 & i & k\end{bmatrix},\boldsymbol{F}=\begin{bmatrix}-1+4i+j-2k & -5-2i-j-2k & 2i-3j\\ 2+2i+4j+k & -3+2i+2j-k & -7-i-2k\\ 1+10i+6j+k & -5+i-j-3k & -7+2i+6k\end{bmatrix}$$

(a) Discuss whether the cyclic solution of Stein equation exists or not. If it exists, find its solution $\boldsymbol{X}$.

(b) Given the quaternion circulant matrix $\boldsymbol{P}=\begin{bmatrix}1 & i & j\\ j & 1 & i\\ i & j & 1\end{bmatrix}\in\mathbf{Q}^{3\times3}$, try to find the optimal approximate solution of problem 2.

Solution. (a) Write the complex representation matrices $L(\boldsymbol{A}),L(\boldsymbol{B}),L(\boldsymbol{F})$ of the $\boldsymbol{A},\boldsymbol{B},\boldsymbol{F}$ by Definition 1, and then write the $\boldsymbol{M},\tilde{\boldsymbol{D}},\boldsymbol{W}$ by (11), Through calculation, shows that $(\boldsymbol{W}\tilde{\boldsymbol{D}})(\boldsymbol{W}\tilde{\boldsymbol{D}})^{+}vec(L(\boldsymbol{F}))=vec(L(\boldsymbol{F}))$, therefore, according to Theorem 1, the cyclic solution of the Stein equation $\boldsymbol{AXB}-\boldsymbol{X}=\boldsymbol{F}$ exists, and the expression of the cyclic solution is

$$\boldsymbol{X}=\frac{1}{4}[\boldsymbol{I}_3,-j\boldsymbol{I}_3]\cdot\left[\boldsymbol{Y}+\boldsymbol{Q}_6\bar{\boldsymbol{Y}}\boldsymbol{Q}_6^{-1}\right]\cdot\begin{bmatrix}\boldsymbol{I}_3\\ j\boldsymbol{I}_3\end{bmatrix}$$

where

$$y=(\boldsymbol{W}\tilde{\boldsymbol{D}})^{+}vec(L(\boldsymbol{F}))+\left[\boldsymbol{I}_{12}-(\boldsymbol{W}\tilde{\boldsymbol{D}})^{+}(\boldsymbol{W}\tilde{\boldsymbol{D}})\right]z,\forall z\in\mathbf{C}^{12\times1}$$
$$y_1=y(1:3),y_3=y(4:6),y_2=y(7:9),y_4=y(10:12).$$

When the free quantity $z=[1,0,0,0,i,0,0,0,1,0,0,0]^T\in\mathbf{C}^{12\times1}$, the error value is

$$\mathrm{Er}(\boldsymbol{X})=\|\boldsymbol{AXB}-\boldsymbol{X}-\boldsymbol{F}\|_F=1.49e-14$$

(b) In the case of $\boldsymbol{P}=\begin{bmatrix}1 & i & j\\ j & 1 & i\\ i & j & 1\end{bmatrix}\in\mathbf{Q}^{3\times3}$, from (14)

$$vec(L(\boldsymbol{P}))=\tilde{\boldsymbol{D}}r,\quad r=[0,i,1,-1,0,0,1,0,0,0,-i,1]^T$$

By Theorem 2

$$\begin{aligned}z_0=[&-1.4270+0.0340i,1.5335+1.3657i,0.1036+1.2215i,0.3044+0.4378i,\\ &-0.7416-0.2012i,-1.0251+0.0185i,-1.1574+0.1987i,-1.9251+0.3128i\\ &0.3828-1.4414i,-0.9442-0.4007i,-0.0589-1.3970i,0.9935+0.7430i]^T\end{aligned}$$

Therefore, the optimal approximate solution of problem 2 is

$$\boldsymbol{X}=\begin{bmatrix}1.10-0.11i+0.13j-0.35k & -1.48-1.84i+1.01j-0.41k & 0.69+0.89i+0.04i+0.47k\\ 0.69+0.89i+0.04i+0.47k & 1.10-0.11i+0.13j-0.35k & -1.48-1.84i+1.01j-0.41k\\ -1.48-1.84i+1.01j-0.41k & 0.69+0.89i+0.04i+0.47k & 1.10-0.11i+0.13j-0.35k\end{bmatrix}$$

The error value is $\mathrm{Er}(\boldsymbol{X})=\|\boldsymbol{X}-\boldsymbol{P}\|_F=3.998$.

It is proved that the results obtained from this example using the parametric transformation method are identical to the results of the complex representation method, and the process is omitted.

7. Summary

It is concluded that the Stein equation is a kind of matrix equation that is widely used, and its cyclic solution is discussed in quaternion field. For problem 1, by using the complex representation of the matrix $\boldsymbol{A},\boldsymbol{B},\boldsymbol{F}$ and the Kronecker product of the matrix, (1) is transformed into an unconstrained cyclic matrix equation equivalently, and the necessary and sufficient condition for the existence of cyclic structure solution of quaternion Stein equation and its expression are obtained. Aiming at problem 2, using the properties of circulant matrix and the formula of minimum norm least square solution, under the condition $\Omega \neq \varnothing$ of cyclic solution problem 1, the best approximation solution with minimum Frobenius norm is obtained with the given quaternion circulant matrix $\boldsymbol{P}$. The findings extend a new type of structural solution of the quaternion Stein equation.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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