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Research Article

On Certain Properties of a Univalent Function Associated with Beta Function

Matthew Olanrewaju Oluwayemi,[1,2] **Sunday Olufemi Olatunji**,[3] **and Temitope O. Ogunlade**[1,2,4]

[1]*Landmark University SDG 4 (Quality Education Research Group), Omu-Aran, Nigeria*
[2]*Department of Mathematics, Landmark University, P.M.B, 1001 Omu-Aran, Nigeria*
[3]*Department of Mathematical Sciences, Federal University of Technology, P.M.B, 704 Akure, Nigeria*
[4]*Department of Mathematics, Ekiti State University, Ado-Ekiti, Nigeria*

Correspondence should be addressed to Matthew Olanrewaju Oluwayemi; oluwayemimatthew@gmail.com

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Beta function has some applications in differential equations and other areas of sciences and engineering where certain definite integrals are used. However, its applications to univalent functions have not been explored based on the available literature. In this work, therefore, the authors defined a univalent function associated with the beta function $B(m, n)$ with $m, n > 0$. Some geometric properties of the function are discussed.

1. Introduction and Preliminaries

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalised by $f(0) = 0$ and $f'(0) = 1$ and let T denote the subclass of S consisting of functions whose nonzero coefficients, in the second on, are negative which is defined by

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k ; a_k > 0. \tag{2}$$

See [1–4].

Quite a number of life problems resulted from differential equations when formulated mathematically. These equations come in the form of partial differential equations (PDE) and ordinary differential equations (ODE). The ODE can also come in different orders such as first order or second order depending on the number of factors. ODEs could be linear equations or nonlinear equations. There are different methods of analytical solutions of the second–order differential equations and the peculiarity of the differential equations matter. For linear ODE, for instance, the most effective method of obtaining the solutions once the variable coefficients are analytic functions is the power series solution. They are called power series methods because the solutions come in certain series forms. The series solution is effective for most differential equations. Special functions such as Bessel equation, Hermite equation, Frobenius equation, hypergeometric equation, Laguerre equation, and Legendre equation are in the form of differential equations. Other special functions such as Chebyshev polynomials, Hermite polynomial, Legendre polynomial, factorial function, gamma function, and beta function are also useful techniques of solving problems in mathematics. Both beta and gamma functions are called the Euler's integrals. A beta function can be in form of polynomial, polar, or the gamma function. It is often referred to as the Eulerian integral of the first kind while the gamma function is the Eulerian integral of the second kind.

The beta function denoted by $B(m, n)$ is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad (3)$$

which converges for $m > 0$ and $n > 0$. There are three cases of beta function which include:

(1) When m is a positive integer

(2) When n is a positive integer

(3) When both m and n are positive integers

Using case 3, we can express a beta function in the form of the gamma function as follows:

$$\begin{aligned} B(m, n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!} \\ &= \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = B(n, m), \end{aligned} \quad (4)$$

which indicates the symmetric property of the beta function [5, 6].

By ratio test, let

$$a_k = \frac{(m-1)!(n-1)!}{(m+n-1)!} \text{ and } a_{k+1} = \frac{n!(m-1)!}{(m+n)!}. \quad (5)$$

Then,

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{n!(m-1)!}{(m+n)!} \times \frac{(m+n-1)!}{(m-1)!(n-1)!} \\ &= \frac{n(n-1)!(m+n-1)!}{(m+n)(m+n-1)!(n-1)!} = \frac{n}{m+n}, \end{aligned} \quad (6)$$

converges if

$$\mathrm{Lim}_{k \longrightarrow \infty} \frac{n}{m+n} < 1. \quad (7)$$

That is, $m > 0$ and diverges for

$$\mathrm{Lim}_{k \longrightarrow \infty} \frac{n}{m+n} > 1. \quad (8)$$

That is, $0 > m$ or $m < 0$ which implies that $B(m, n)$ converges only for case 3 and diverges for either case 1 or case 2.

Suppose $\rho = n/m + n$, then the radius of convergence $1/\rho = m + n/n = m/n + 1$ or $1 + m/n$. There are three possible cases of the radius of convergence:

(1) When $m = n$; then, the radius of convergence is 2

(2) When $m > n$; then, the radius of convergence is greater than 2

(3) When $m < n$; then, the radius of convergence is less than 2

In general, we conclude that the radius of convergence, r is such that $r \in (1, \mu]$ where $\mu \geq 2$. Then, we consider the interval of convergence is $[-2, 2]$. We consider a case of n, $m \in \mathbb{N}$.

Suppose m and n are distinct values, that is, $m \neq n$. Then, we have that

$$\begin{aligned} f_{m,n}(z) &= z + \sum_{k=2}^{\infty} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right) z^k \\ &= z + \sum_{k=2}^{\infty} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) z^k ; m, n > 0, \end{aligned} \quad (9)$$

is a normalised univalent function which belongs to class S with beta function as its coefficients. Hence,

$$\begin{aligned} f_{m,n}(z) &= z + \frac{1}{12}z^2 + \frac{1}{60}z^3 + \frac{1}{280}z^4 + \frac{1}{1260}z^5 \\ &+ \frac{1}{5544}z^6 + \frac{1}{48048}z^7 + \frac{1}{25740}z^8 \\ &+ \frac{1}{437580}z^9 + \frac{1}{461890}z^{10} + \cdots. \end{aligned} \quad (10)$$

If $m = n$; then, equation (9) reduces to

$$f_{m,n}(z) = z + \sum_{k=2}^{\infty} \left\{ \frac{[\Gamma(n)]^2}{\Gamma(2n)} \right\} z^k = z + \sum_{k=2}^{\infty} \left\{ \frac{[(n-1)!]^2}{(2n-1)!} \right\} z^k ; n > 0. \quad (11)$$

From equations (9) and (11), we have that $a_k = B(m, n)$ and $a_k = B(n, n)$ as the coefficients of (9) and (11), respectively. In addition,

$$F_{m,n}(z) = 2z - f_{m,n}(z) = z - \sum_{k=2}^{\infty} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) z^k, \quad (12)$$

belongs to the class T with negative coefficients so that $f_{m,n}(z) \in S$ and $F_{m,n}(z) \in T$.

Let

$$F_{m,n}(z) = z - \sum_{k=2}^{\infty} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) z^k. \quad (13)$$

Equations (9) an (13) are special case of (1) and (2), respectively.

Given $m, n > 0$, let

$$a_k = \frac{(m-1)!(n-1)!}{(m+n-1)!} ; k \geq 1, \quad (14)$$

where $m \neq n$. Then, $a_{1,2} = 0!1!/2! = 1/2$. But rather, we consider the case where $m = n$ and give the coefficients as follows:

$$a_1 = B(1,1) = \frac{0!0!}{1!} = 1, \tag{15}$$

$$a_2 = B(2,2) = \frac{1!1!}{3!} = \frac{1}{6}, \tag{16}$$

$$a_3 = B(3,3) = \frac{2!2!}{5!} = \frac{2.2!}{5.4.3.2!} = \frac{1}{5.3.2} = \frac{1}{30}, \tag{17}$$

$$a_4 = B(4,4) = \frac{3!3!}{7!} = \frac{6.3!}{7.6.5.4.3!} = \frac{1}{7.5.4} = \frac{1}{140}, \tag{18}$$

$$a_5 = B(5,5) = \frac{4!4!}{9!} = \frac{24.4!}{9.8.7.6.5.4!} = \frac{1}{7.6.5.3} = \frac{1}{630}, \tag{19}$$

$$\begin{aligned} a_6 = B(6,6) &= \frac{5!5!}{11!} = \frac{120.5!}{11.10.9.8.7.6.5!} \\ &= \frac{1}{11.9.8.7.2} = \frac{1}{11088}, \end{aligned} \tag{20}$$

$$\begin{aligned} a_7 = B(7,7) &= \frac{6!6!}{13!} = \frac{720.6!}{13.12.11.10.9.8.7.6!} \\ &= \frac{1}{13.11.8.7.4.3} = \frac{1}{96096}, \end{aligned} \tag{21}$$

$$\begin{aligned} a_8 = B(8,8) &= \frac{7!7!}{15!} = \frac{5040.7!}{15.14.13.10.9.8.7!} \\ &= \frac{1}{15.13.8.3} = \frac{1}{51480}, \end{aligned} \tag{22}$$

$$\begin{aligned} a_9 = B(9,9) &= \frac{8!8!}{17!} = \frac{40320.8!}{17.16.15.14.13.12.11.10.9.8!} \\ &= \frac{1}{17.13.11.10.9} = \frac{1}{240669}, \end{aligned} \tag{23}$$

$$\begin{aligned} a_{10} = B(10,10) &= \frac{9!9!}{19!} = \frac{120.5!}{11.10.9.8.7.6.5!} \\ &= \frac{1}{11.9.8.7.2} = \frac{1}{923780}. \end{aligned} \tag{24}$$

If $a_k = a_{m,n} = B(m,n)$, the table below shows the possible 1st – 10thcoefficients of a normalised univalent function associated with the Eulerian integral of the first kind (Table 1). It is obvious from the below table that $|a_k| \le 1$ for all $k \ge 1$.

Hence, equation (9) becomes

$$f_{1,1}(z) = z + z^2 + z^3 + \cdots = z + \sum_{k=2}^{\infty} z^k. \tag{25}$$

Consider the special case of (9) with $a_k = a_{m,n}$ such that $m = n$, we have that

$$\begin{aligned} f_{n,n}(z) = z &+ \frac{1}{6}z^2 + \frac{1}{30}z^3 + \frac{1}{140}z^4 + \frac{1}{630}z^5 \\ &+ \frac{1}{11088}z^6 + \frac{1}{96096}z^7 + \frac{1}{51480}z^8 \\ &+ \frac{1}{218790}z^9 + \frac{1}{923780}z^{10} + \cdots, \end{aligned} \tag{26}$$

$$f_{n,n}(z) = z + \sum_{k=2}^{\infty} \left\{ \frac{[(n-1)!]^2}{(2n-1)!} \right\} z^k ; n > 0, \tag{27}$$

$$F_{n,n}(z) = z - \sum_{k=2}^{\infty} \left\{ \frac{[(n-1)!]^2}{(2n-1)!} \right\} z^k ; n > 0. \tag{28}$$

In [2, 7], the authors introduced a generalised multiplier transformation as a tool to define a subclass of univalent functions. The application of the multiplier transformation was considered in [8]. One of the interests in complex analysis is that every analytic function has a power series expansion. This further encourages the use of a normalised univalent functions of the form (1); and in this work, we consider a normalised univalent function whose coefficients are in terms of the Eulerian integral of the first kind. Altunhan and Eker in [9] earlier explored subclasses of univalent functions with coefficients in Poisson distribution series while many other authors have also considered various forms of special functions, see [4, 10, 11] for examples.

2. Main Results

Theorem 1. *Let the function $f(z)$ be defined by (13). Then,*

$$\sum_{k=2}^{\infty} k \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) \le 1. \tag{29}$$

Proof. Let $f(z) \in T$, then

$$f'(z) = 1 - \sum_{k=2}^{\infty} k \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) z^{k-1}, \tag{30}$$

is nonnegative. That is,

$$\begin{aligned} f'(r) = 1 - \sum_{k=2}^{\infty} k \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) r^{k-1} \\ > 0 ; (|z| = r) \text{ and } k \ge 2. \end{aligned} \tag{31}$$

The sequence $a_k = \{k((m-1)!(n-1)!/(m+n-1)!)\}_{k=2}^{\infty}$ is a monotonic decreasing sequence such that $\sum_{k=2}^{\infty} a_k \longrightarrow 1$ as $k \longrightarrow \infty$ and by letting $|z| = r \longrightarrow 1$, we have that

$$f'(r) = 1 - \sum_{k=2}^{\infty} k \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) > 0. \tag{32}$$

Since $m, n > 0$. Then,

$$\sum_{k=2}^{\infty} k \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) < 1, \tag{33}$$

for $k \ge 2$. □

Table 1

	First ten coefficients of beta function		
$a_k = a_{m,n}$	Case 1: $(m < n)$	Case 2: $(m > n)$	Case 3: $(m = n)$
$a_1 = a_{m,n}$	1	1	$a_{1,1} = 1$
$a_2 = a_{m,n}$	$a_{2,3} = 1/12$	$a_{3,2} = 1/12$	$a_{2,2} = 1/6$
$a_3 = a_{m,n}$	$a_{3,4} = 1/60$	$a_{4,3} = 1/60$	$a_{3,3} = 1/30$
$a_4 = a_{m,n}$	$a_{4,5} = 1/280$	$a_{5,4} = 1/280$	$a_{4,4} = 1/140$
$a_5 = a_{m,n}$	$a_{5,6} = 1/1260$	$a_{6,5} = 1/1260$	$a_{5,5} = \frac{1}{630}$
$a_6 = a_{m,n}$	$a_{6,7} = 1/5544$	$a_{7,6} = 1/5544$	$a_{6,6} = \frac{1}{11088}$
$a_7 = a_{m,n}$	$a_{7,8} = 1/48048$	$a_{8,7} = 1/48048$	$a_{7,7} = 1/96096$
$a_8 = a_{m,n}$	$a_{8,9} = 1/25740$	$a_{9,8} = 1/25740$	$a_{8,8} = 1/51480$
$a_9 = a_{m,n}$	$a_{9,10} = 1/437580$	$a_{10,9} = 1/48048$	$a_{9,9} = 1/218790$
$a_{10} = a_{m,n}$	$a_{10,11} = 1/461890$	$a_{11,10} = 1/461890$	$a_{10,10} = 1/923780$

Theorem 2. *Let the function*

$$F_{m,n} = z - \sum_{k=2}^{\infty} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) z^k, \tag{34}$$

belong to class T. Then, the function $G_{m,n}(z)$ defined as

$$G_{m,n}(z) = \sum_{\mu=1}^{j} \Gamma_\mu F_{m,n}(z)\,;\, j \in \mathbb{N} \text{ and } \Gamma_\mu \geq 0, \tag{35}$$

also belongs to class T with

$$\sum_{\mu=1}^{j} \Gamma_\mu = 1. \tag{36}$$

Proof. By definition,

$$G_{m,n}(z) = \sum_{\mu=1}^{j} \left\{ \sum_{k=2}^{\infty} \Gamma_\mu \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) \right\} z^k. \tag{37}$$

Since $F_{m,n}$ belongs to class T for every $m, n > 0$ and $k \geq 2$. Then,

$$\sum_{k=2}^{\infty} k \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) \leq 1, \tag{38}$$

for every $m, n > 0$ and $k > 2$. Thus,

$$\begin{aligned} &\sum_{\mu=1}^{j} k \left\{ \sum_{k=2}^{\infty} \Gamma_\mu \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) \right\} \\ &= \sum_{\mu=1}^{j} \Gamma_\mu \left\{ \sum_{k=2}^{\infty} k \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) \right\} \leq \sum_{\mu=1}^{j} \Gamma_\mu = 1, \end{aligned} \tag{39}$$

which completes the proof. □

Theorem 3. *Let the function*

$$F_{m,n} = z - \sum_{k=2}^{\infty} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) z^k, \tag{40}$$

be in the class T and let ρ be a real number such that $\rho > -1$. Then, the function

$$f_{m,n} = \frac{\rho+1}{z^\rho} \int_0^z s^{\rho-1} f(s) ds, \tag{41}$$

also belongs to the class T.

Proof. From the definition, we can express $F_{m,n}$ as follows

$$F_{m,n} = \sum_{k=2}^{\infty} c_k z^k, \tag{42}$$

where

$$c_k = \frac{\rho+1}{\rho+k} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right). \tag{43}$$

Thus,

$$\begin{aligned} \sum_{k=2}^{\infty} k c_k &= \sum_{k=2}^{\infty} k \left\{ \frac{\rho+1}{\rho+k} \right\} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) \\ &\leq \sum_{k=2}^{\infty} k \left(\frac{(m-1)!(n-1)!}{(m+n-1)!} \right) \leq 1, \end{aligned} \tag{44}$$

which completes the proof. □

Theorem 4. *Let ρ be a real number such that $\rho > -1$. If $F_{m,n} \in T$, then, the function $f_{m,n}$ defined by (41) is univalent in $|z| < R$ where*

$$R = inf_k \left(\frac{\rho + 1}{\rho + k}\right)^{1/k-1} k > 2. \quad (45)$$

Proof. Let $f_{m,n}$ be defined by (13). Then, from (41), it follows that

$$f_{m,n} = \frac{z^{1-\rho}\left[z^{\rho} f_{m,n}\right]'}{\rho + 1} = z - \sum_{k=2}^{\infty} \frac{\rho + k}{\rho + 1} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right) z^k ; \rho > -1. \quad (46)$$

We now need to show that $|f_{m,n} - 1| < 1$ in $|z| < R$ where

$$\left|f_{m,n} - 1\right| \le \sum_{k=2}^{\infty} k \left\{\frac{\rho + k}{\rho + 1}\right\} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right) \left|z^{k-1}\right|. \quad (47)$$

Hence, $|f_{m,n} - 1| < 1$ provided

$$\sum_{k=2}^{\infty} \frac{k(\rho + k)}{\rho + 1} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right) |z|^{k-1} < 1. \quad (48)$$

From (29), we have that (48) is satisfied if

$$\frac{k(\rho + k)}{\rho + 1} |z|^{k-1} \le 1, \quad (49)$$

which implies that

$$|z| \le \inf_k \left(\frac{\rho + 1}{\rho + k}\right)^{1/k-1} (k \ge 2), \quad (50)$$

as required. Thus, the function $f_{m,m}(z)$ is univalent in $|z| < R$ with the best estimate

$$f_{m,n} = z - \left\{\frac{\rho + 1}{k(\rho + k)}\right\} z^k (k > 2). \quad (51)$$

□

3. Application of Fractional Calculus

We now consider an application of fractional calculus earlier introduced by [12, 13] to the univalent function $F_{m,n}$ $(m, n > 0)$ involving beta function $B(m, n)$ as follows.

Definition 5. The fractional integral of order $\sigma > 0$ for a function $f_{m,n}(z)$ is defined as

$$D_z^{-\sigma} f_{m,n}(z) = \frac{1}{\Gamma(\sigma)} \int_0^z \frac{f(s)ds}{(z-s)^{1-\sigma}}, \quad (52)$$

where $0 \le \sigma < 1$, $f_{m,n}(z)$ is an analytic function in a simply connected region of the $z-$ plane containing the origin and the multiplicity of $(z-s)^{\sigma-1}$ is removed by requiring $\log (z - \eta)$ to be real when $(z - s) > 0$.

Definition 6. The fractional integral of order $\sigma > 0$ for a function $f_{m,n}(z)$ is defined by

$$D_z^{\sigma} f_{m,n}(z) = \frac{1}{\Gamma(1-\sigma)} \frac{d}{dz} \int_0^z \frac{f(s)ds}{(z-s)^{\sigma}}, \quad (53)$$

where $\sigma > 0$, $f_{m,n}(z)$ is an analytic function in a simply connected region of the $z-$ plane containing the origin and the multiplicity of $(z-s)^{-\sigma}$ is removed by requiring $\log (z - \eta)$ to be real when $(z - s) > 0$.

Definition 7. Under the hypotheses of Definition 6., the fractional derivative of order $(k + \sigma)$ is defined by

$$D_z^{k+\sigma} f_{m,n}(z) = \frac{d^k}{dz^k} D_z^{\sigma} f_{m,n}(z), \quad (54)$$

where $0 < \sigma < 1$ and $k \in \mathbb{N}_0$.

Theorem 8. *Let $f_{m,n}(z) \in T$. Then,*

$$\frac{|z|^{1+\sigma}}{\Gamma(2+\sigma)} \left[1 - \frac{\Gamma(3)\Gamma(\sigma+1)}{2(\Gamma(3+\sigma))} |z|\right] \le \left|D_z^{-\sigma} f_{m,n}(z)\right| \le \frac{|z|^{1+\sigma}}{\Gamma(2+\sigma)} \left[1 + \frac{\Gamma(3)\Gamma(\sigma+1)}{2(\Gamma(3+\sigma))} |z|\right]. \quad (55)$$

Proof. Using the above definition and upon simple calculation, we have that

$$\begin{aligned} &\Gamma(2+\sigma) z^{-\sigma} D_z^{-\sigma} f_{m,n}(z) \\ &\quad = z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(\sigma+1)}{\Gamma(k+\sigma+1)} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right) z^k \\ &\quad = z - \sum_{k=2}^{\infty} \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right) z^k, \end{aligned} \quad (56)$$

setting

$$\frac{\Gamma(k+1)\Gamma(\sigma+1)}{\Gamma(k+\sigma+1)} = \Phi(k), \quad (57)$$

$$a_{m,n} = \left(\frac{(m-1)!(n-1)!}{(m+n-1)!}\right) = a_k, \quad (58)$$

so that $k(m, n)$ is a function of m and n such that $m, n \ge 0$. It is obvious that $\Phi(k)$ is a strictly monotonic decreasing sequence since $\Gamma(k+1)(\sigma+1) < \Gamma(k+\sigma+1)$ for all values of $k \ge 2$. It thus follows that

$$0 < \Phi(k) \le \Phi(2) = \frac{\Gamma(3)\Gamma(\sigma+1)}{\Gamma(3+\sigma)}. \quad (59)$$

Thus,

$$\begin{aligned} &\left|\Gamma(2+\sigma)z^{-\sigma}D_z^{-\sigma}f_{m,n}(z)\right| \\ &\quad\geq \Phi(2)|z|^2\sum_{k=2}^{\infty}|a_k| \geq |z| - \frac{1}{2}\Phi(2)|z|^2 \\ &\quad= z - \frac{\Gamma(3)\Gamma(\sigma+1)}{2\Gamma(3+\sigma)}|z|^2, \end{aligned} \tag{60}$$

$$\left|\Gamma(2+\sigma)z^{-\sigma}D_z^{-\sigma}f_{m,n}(z)\right| \leq z + \frac{\Gamma(3)\Gamma(\sigma+1)}{2\Gamma(3+\sigma)}|z|^2. \tag{61}$$

Hence,

$$\begin{aligned} &\frac{|z|^{1+\sigma}}{\Gamma(2+\sigma)}\left[1 - \frac{\Gamma(3)\Gamma(\sigma+1)}{2(\Gamma(3+\sigma))}|z|\right] \\ &\quad\leq \left|D_z^{-\sigma}f_{m,n}(z)\right| \leq \frac{|z|^{1+\sigma}}{\Gamma(2+\sigma)}\left[1 + \frac{\Gamma(3)\Gamma(\sigma+1)}{2(\Gamma(3+\sigma))}|z|\right], \end{aligned} \tag{62}$$

as required. □

4. Conclusion

In this study, a certain univalent function associated with the Eulerian integral of the first kind, otherwise known as beta function, is established and some of the geometric properties of the functions discussed. The study opens up a new direction of investigation in geometric function theory (GFT).

Data Availability

No data was used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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