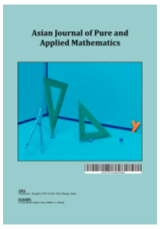

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A Note on Soft Regular and Soft Inverse Semigroups

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Authors' contributions

This work was carried out in collaboration among all authors. Author UC conceptualized and designed the study, contributed some results obtained in the work and wrote the first draft of the manuscript. Author OMC reviewed and organized the final manuscript. Author NRU contributed some results in the work as well as edited the first draft. All authors read and approved the final manuscript.

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Abstract

The concept of soft regular semigroups has already been introduced in literature by Ali et al. in 2009. Regular semigroups can be considered as core semigroups since it shares the same properties with groups. In this paper, we present some properties of soft regular semigroups and introduce the concept of soft inverse semigroups. Consequently, we present some properties of soft inverse semigroups.

Keywords: Soft sets; soft semigroups; soft regular semigroups; soft inverse semigroups.

Mathematics Subject Classification: *20M10.*

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1 Introduction

Renowned British Mathematician, Sir Andrew Wiles, famous for having proved Fermat's last theorem in 1995 once said "The definition of a good mathematical problem is the mathematics it generates rather than the problem itself." No doubts uncertainty is one good mathematical problem which has generated much good mathematics due to its complex nature.

Many scientists have proposed so many theories that try to model uncertainty, famous of which is the fuzzy set theory by Zadeh [1]. Molodstov [2] on the other hand came up with soft set theory in 1999, a theory for which aside mathematics has generated a wide field of study in so many disciplines such as medical sciences, social sciences, engineering, economics and so on. His theory has been studied and applied in so many branches of mathematics, thus giving a new face to the concept of completeness and precision in mathematics [see 3-9].

In 2007, Aktas and Cagman [3], introduced soft groups to herald the emergence of soft algebraic structures. Jun et al., [10] studied soft ideals and idealistic soft BCK/BCI Algebras. Soft rings were introduced by Acar et al. [11]. Feng et al. [12] studied soft semirings and its several related notions to establish a connection between soft sets and semirings. Ali et al. [4] defined soft semigroups and considered soft ideals, soft quasi ideals and soft bi-ideals over a given semigroup $\mathcal{S}$. They went on to characterize regular semigroups using soft ideals and as a consequence, introduced soft regular semigroups.

Regular semigroups can be considered as core semigroups since it shares the same properties with groups. In fact, a group is a regular semigroup with a unique idempotent. The notion was introduced by Green [13] in 1951. In this paper we add to the work of Ali et al. [4], by giving some more properties of soft regular semigroups. Also, Ndubuisi et al. in [14] characterized soft sets as soft semigroups. They showed that some semigroup properties are embedded in soft sets.

Inverse Semigroups on the other hand were introduced first by Vagner [15] in 1952, and Preston [16] in 1954, independently. Vagner called them "generalized groups" and for many years, the nomenclature was standard in the Russian literature. In both cases, the origin of the idea was the study of semigroups of partial one-to-one mappings of a set. One of its earliest results was the representation theorem (which is analogous to the cayley's theorem in group theory) to the effect that every inverse semigroup has a faithful representation as an inverse semigroup of partial one-to-one mappings. This germane result later became known as the Vagner-Preston theorem.

The theory of inverse semigroups has many features in common with the theory of groups, but there are some important differences, amongst which is the natural partial order that exists in each inverse semigroup. The works of McAlister [17], Ndubuisi [18,19] and others on E- Unitary inverse semigroups moved the study of inverse semigroups to a new phase.

In this paper, we introduce and study the basic notion of soft inverse semigroups which extends the notion of inverse semigroups to include the algebraic structures of soft sets. We also study some of its properties.

2 Preliminaries

Definition 2.1 [20]: An element $\mathfrak{x}$ of a semigroup $\mathcal{S}$ is said to be *regular* if there exists an element $\mathfrak{y} \in \mathcal{S}$ such that

$$\mathfrak{x}\mathfrak{y}\mathfrak{x} = \mathfrak{x}. \qquad (*)$$

A semigroup $\mathcal{S}$ is called a *regular semigroup* provided every element in $\mathcal{S}$ is regular.

$(*)$ is known as the regularity condition. It is equivalent to the seemingly stronger condition that; for all $\mathfrak{x} \in \mathcal{S}$ there exists $\mathfrak{y} \in \mathcal{S}$ such that

$$\mathfrak{x}\mathfrak{y}\mathfrak{x} = \mathfrak{x} \text{ and } \mathfrak{y}\mathfrak{x}\mathfrak{y} = \mathfrak{y}. \qquad (**)$$

The equivalence of these conditions was shown in [20]. An element $\mathfrak{y}$ satisfying $(**)$ is called the *inverse* of $\mathfrak{x}$.

It is important to note that in a regular semigroup an element could have more than one inverse. This is expressed clearly by the following result:

Lemma 2.2 [21]: If $\mathfrak{x}$ is a regular element of a semigroup $\mathcal{S}$, say $\mathfrak{x}\mathfrak{y}\mathfrak{x} = \mathfrak{x}$ with $\mathfrak{y}$ in $\mathcal{S}$, then $\mathfrak{x}$ has at least one inverse in $\mathcal{S}$, in particular $\mathfrak{y}\mathfrak{x}\mathfrak{y}$.

A regular semigroup must contain idempotent elements. This follows immediately from $(*)$, as $\mathfrak{x}\mathfrak{y}$ and $\mathfrak{y}\mathfrak{x}$ are idempotents.

Example 2.3 [20]:

i) Every group G is a regular semigroup since for each $a \in \mathcal{S}$ there exists a^{-1} such that

$$aa^{-1}a = a \text{ and } a^{-1}aa^{-1} = a^{-1},$$

where a^{-1} is the inverse of a.

ii) Let $\mathcal{S} = \{a, b, k, p\}$ be a semigroup given by the Table 1.

Table 1. Regular semigroup

	a	b	k	p
a	a	b	a	a
b	a	b	a	b
k	a	p	k	p
p	a	a	k	p

Then it can be seen that $\mathcal{S}$ is a regular semigroup.

The next definition is pivotal;

Definition 2.4 [20]: Let $\mathcal{S}$ be a regular semigroup. If for each $a \in \mathcal{S}$, there exists a unique element $b \in \mathcal{S}$ in the sense that $aba = a$ and $bab = b$, then $\mathcal{S}$ is called an *inverse semigroup*.

Example 2.5 [20]: (i) Given $\mathcal{S} = \{\mathfrak{h}, \mathfrak{q}, \mathfrak{u}, \mathfrak{s}, \mathfrak{w}\}$, with the multiplication Table 2.

Table 2. Inverse semigroup

	$\mathfrak{h}$	$\mathfrak{q}$	$\mathfrak{u}$	$\mathfrak{s}$	$\mathfrak{w}$
$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$
$\mathfrak{q}$	$\mathfrak{h}$	$\mathfrak{q}$	$\mathfrak{u}$	$\mathfrak{h}$	$\mathfrak{h}$
$\mathfrak{u}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{q}$	$\mathfrak{u}$
$\mathfrak{s}$	$\mathfrak{h}$	$\mathfrak{s}$	$\mathfrak{w}$	$\mathfrak{h}$	$\mathfrak{h}$
$\mathfrak{w}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{s}$	$\mathfrak{w}$

It is clear that $\mathcal{S}$ is an inverse semigroup.

(ii) Every group is an inverse semigroup, so also is every semilattice.

Theorem 2.6 [22]: A regular semigroup in which idempotents commute is an inverse semigroup, that is, every element has a unique inverse.

Lemma 2.7 [21]: For any elements a, b of an inverse semigroup $\mathcal{S}$, we have $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Definition 2.8 [2]: A pair $(\mathfrak{F}, \mathcal{A})$ is called a *soft set* over $\mathfrak{U}$. Here, $\mathfrak{F}: \mathcal{A} \to \mathfrak{P}(\mathfrak{U})$ is a mapping with $\mathcal{A} \subseteq \mathfrak{E}$, $\mathfrak{E}$ is the set of parameters that define the elements of the universe set $\mathfrak{U}$ and $\mathfrak{P}(\mathfrak{U})$ is the set of all subsets of $\mathfrak{U}$.

For any $\mathfrak{e} \in \mathcal{A}$, $\mathfrak{F}(\mathcal{A})$ is known as the set of $\mathfrak{e}$ – approximate elements of the soft set $(\mathfrak{F}, \mathcal{A})$. It is important to note that soft sets are clearly not sets.

Several examples were considered by Molodtsov in [2]. More examples were also given in [6]. Soft sets can also be represented in tabular form. Refer to [23] and [6].

Definition 2.9 [4]: Let $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$be soft sets over $\mathfrak{U}$. Then $(\mathfrak{G}, \mathfrak{B})$ is a *soft subset* of $(\mathfrak{F}, \mathcal{A})$ if $\mathfrak{B} \subseteq \mathcal{A}$ and $\mathfrak{G}(b) \subseteq \mathfrak{F}(b) \forall\, b \in \mathfrak{B}$.

Definition 2.10 [4]: If $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$are soft sets over $\mathfrak{U}$, then the *basic intersection* of $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$, "$(\mathfrak{F}, \mathcal{A})$ AND $(\mathfrak{G}, \mathfrak{B})$" is denoted by $(\mathfrak{F}, \mathcal{A}) \wedge (\mathfrak{G}, \mathfrak{B})$ and is defined by $(\mathfrak{F}, \mathcal{A}) \wedge (\mathfrak{G}, \mathfrak{B}) = (\mathcal{H}, \mathcal{A} \times \mathfrak{B})$, where $\mathcal{H}(\mathfrak{a}, b) = \mathfrak{F}(\mathfrak{a}) \cap \mathfrak{G}(b)$ for all $(\mathfrak{a}, b) \in \mathcal{A} \times \mathfrak{B}$.

Definition 2.11 [4]: If $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$are soft sets over $\mathfrak{U}$, then the *basic union* of $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$, "$(\mathfrak{F}, \mathcal{A})$OR $(\mathfrak{G}, \mathfrak{B})$" is denoted by $(\mathfrak{F}, \mathcal{A}) \vee (\mathfrak{G}, \mathfrak{B})$ and is defined by $(\mathfrak{F}, \mathcal{A}) \vee (\mathfrak{G}, \mathfrak{B}) = (\mathcal{H}, \mathcal{A} \times \mathfrak{B})$, where $\mathcal{H}(\mathfrak{a}, b) = \mathfrak{F}(\mathfrak{a}) \cup \mathfrak{G}(b)$ for all $(\mathfrak{a}, b) \in \mathcal{A} \times \mathfrak{B}$.

The terminology 'basic' was first used by Pei and Miao [24].

Definition 2.12 [4]: If $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$are soft sets over $\mathfrak{U}$,then their *extended intersection*, denoted by $(\mathfrak{F}, \mathcal{A}) \cap_{\mathrm{Ex}} \mathcal{F}(\mathfrak{G}, \mathfrak{B})$ is the soft set $(\mathcal{H}, \mathfrak{C})$, where $\mathfrak{C} = \mathcal{A} \cup \mathfrak{B}$ and such that for all $b \in \mathfrak{C}$, it is either $\mathcal{H}(b) = \mathfrak{F}(b)$ if $b \in \mathcal{A} \backslash \mathfrak{B}$ OR $\mathcal{H}(b) = \mathfrak{G}(b)$ if $b \in \mathfrak{B} \backslash \mathcal{A}$ OR $\mathcal{H}(b) = \mathfrak{F}(b) \cap \mathfrak{G}(b)$ if $b \in \mathcal{A} \cap \mathfrak{B}$.

Definition 2.13 [4]: If $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ are soft sets over $\mathfrak{U}$, then their *restricted intersection*, denoted by $(\mathfrak{F}, \mathcal{A}) \cap_{\mathfrak{R}} (\mathfrak{G}, \mathfrak{B})$ is the soft set $(\mathcal{H}, \mathfrak{C})$, where $\mathfrak{C} = \mathcal{A} \cap \mathfrak{B}$ and $\mathcal{H}(b) = \mathfrak{F}(b) \cap \mathfrak{G}(b)$ for all $b \in \mathfrak{C}$.

Definition 2.14 [4]: If $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$are soft sets over $\mathfrak{U}$, then their *extended union*, denoted by $(\mathfrak{F}, \mathcal{A}) \cup_{\mathrm{Ex}} \mathcal{F}(\mathfrak{G}, \mathfrak{B})$ is the soft set $(\mathcal{H}, \mathfrak{C})$, where $\mathfrak{C} = \mathcal{A} \cup \mathfrak{B}$ and such that for all $b \in \mathfrak{C}$, it is either $\mathcal{H}(b) = \mathfrak{F}(b)$ if $b \in \mathcal{A} \backslash \mathfrak{B}$OR $\mathcal{H}(b) = \mathfrak{G}(b)$ if $b \in \mathfrak{B} \backslash \mathcal{A}$ OR $\mathcal{H}(b) = \mathfrak{F}(b) \cup \mathfrak{G}(b)$ if $b \in \mathcal{A} \cap \mathfrak{B}$.

Definition 2.15 [4]: If $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$are soft sets over $\mathfrak{U}$, then their *restricted union*, denoted by $(\mathfrak{F}, \mathcal{A}) \cup_{\mathfrak{R}} (\mathfrak{G}, \mathfrak{B})$ is the soft set $(\mathcal{H}, \mathfrak{C})$, where $\mathfrak{C} = \mathcal{A} \cap \mathfrak{B}$ and $\mathcal{H}(b) = \mathfrak{F}(b) \cup \mathfrak{G}(b)$ for all $b \in \mathfrak{C}$.

Definition 2.16 [4]: A soft set $(\mathfrak{F}, \mathcal{A})$ over $\mathfrak{U}$ is called a *soft set with cover* if $\bigcup_{b \in \mathcal{A}} \mathfrak{F}(b) = \mathfrak{U}$.

Definition 2.17 [4]: Let $\mathcal{S}$ be a semigroup and let $(\mathfrak{F}, \mathcal{A})$ be a soft set over $\mathcal{S}$. Then $(\mathfrak{F}, \mathcal{A})$ is called a *soft semigroup* over $\mathcal{S}$ if and only if for every $b \in \mathcal{A}$, $\mathfrak{F}(b) \neq \emptyset$ is a subsemigroup of $\mathcal{S}$, where $\mathcal{A} \subseteq \mathfrak{E}$, $\mathfrak{E}$ being the set of parameters and $\mathfrak{F}$ a mapping such that $\mathfrak{F}: \mathcal{A} \to \mathfrak{P}(\mathcal{S})$, with $\mathfrak{P}(\mathcal{S})$ being the power set of $\mathcal{S}$.

Definition 2.18 [4]: Let $\mathcal{S}$ be a semigroup and let $(\mathfrak{F}, \mathcal{A})$ be a soft set over $\mathcal{S}$. Then $(\mathfrak{F}, \mathcal{A})$ is called a *soft regular semigroup* over $\mathcal{S}$ if for every $\mathscr{b} \in \mathcal{A}$, $\mathfrak{F}(\mathscr{b})$ is a regular subsemigroup of $\mathcal{S}$.

Example 2.19: Let $\mathcal{S} = \{\mathfrak{w}, \mathfrak{d}, \mathfrak{z}, \mathfrak{x}\}$ be a semigroup given by the Table 3.

Table 3. Soft regular semigroup over $\mathcal{S}$

	$\mathfrak{w}$	$\mathfrak{d}$	$\mathfrak{z}$	$\mathfrak{x}$
$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{x}$
$\mathfrak{d}$	$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{w}$
$\mathfrak{z}$	$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{w}$
$\mathfrak{x}$	$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{w}$	$\mathfrak{x}$

It can be seen from Table 3 that $\mathcal{S}$ is not a regular semigroup. Let $\mathcal{A} = \{\mathscr{r}, \mathscr{q}\}$ be a set of parameters such that $\mathfrak{F}(\mathscr{r}) = \{\mathfrak{w}, \mathfrak{x}\}$ and $\mathfrak{F}(\mathscr{q}) = \{\mathfrak{w}\}$. Since, $\mathfrak{F}(\mathscr{r})$ and $\mathfrak{F}(\mathscr{q})$ are both regular subsemigroups of $\mathcal{S}$,$(\mathfrak{F}, \mathcal{A})$ is a soft regular semigroup over $\mathcal{S}$.

This example goes to tell that the regularity of a soft semigroup does not imply regularity of the semigroup.

3 Main Results

We begin this section by presenting some results on soft regular semigroups. We first present the following definition:

Definition 3.1: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be two soft regular semigroups over $\mathcal{S}$. If $\mathcal{A} \subseteq \mathfrak{B}$ and for all $\mathscr{b} \in \mathcal{A}$, $\mathfrak{F}(\mathscr{b})$ is a subsemigroup of $\mathfrak{G}(\mathscr{b})$, then $(\mathfrak{F}, \mathcal{A})$ is called a *soft regular subsemigroup* of $(\mathfrak{G}, \mathfrak{B})$.

Lemma 3.2: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$be soft regular semigroups over $\mathcal{S}$ such that $\mathcal{A} \cap \mathfrak{B} = \emptyset$. Then their *extended union*,$(\mathfrak{F}, \mathcal{A}) \cup_{\mathrm{E}\mathbb{x}} \mathcal{F}(\mathfrak{G}, \mathfrak{B})$ is also a soft regular semigroup over $\mathcal{S}$.

Proof: Since $(\mathfrak{F}, \mathcal{A}) \cup_{\mathrm{E}\mathbb{x}} \mathcal{F}(\mathfrak{G}, \mathfrak{B}) = (\mathcal{H}, \mathfrak{C})$ and $\mathcal{A} \cap \mathfrak{B} = \emptyset$, for all $\mathscr{b} \in \mathfrak{C} = \mathcal{A} \cup \mathfrak{B}$, it is either $\mathscr{b} \in \mathcal{A} \backslash \mathfrak{B}$ or $\mathscr{b} \in \mathfrak{B} \backslash \mathcal{A}$. If $\mathscr{b} \in \mathcal{A} \backslash \mathfrak{B}$ then $\mathcal{H}(\mathscr{b}) = \mathfrak{F}(\mathscr{b})$ and if $\mathscr{b} \in \mathfrak{B} \backslash \mathcal{A}$, then $\mathcal{H}(\mathscr{b}) = \mathfrak{G}(\mathscr{b})$. In both cases, $\mathcal{H}(\mathscr{b})$ is a regular subsemigroup of $\mathcal{S}$. Therefore, $(\mathcal{H}, \mathfrak{C})$ is a soft regular semigroup over $\mathcal{S}$.

Lemma 3.3: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be soft regular semigroups over $\mathcal{S}$ such that $\mathcal{A} \cap \mathfrak{B} = \emptyset$. Then their *extended intersection*, $(\mathfrak{F}, \mathcal{A}) \cap_{\mathrm{E}\mathbb{x}} \mathcal{F}(\mathfrak{G}, \mathfrak{B})$ is also a soft regular semigroup over $\mathcal{S}$.

Proof: Since $(\mathfrak{F}, \mathcal{A}) \cap_{\mathrm{E}\mathbb{x}} \mathcal{F}(\mathfrak{G}, \mathfrak{B}) = (\mathcal{H}, \mathfrak{C})$, where $\mathfrak{C} = \mathcal{A} \cup \mathfrak{B}$, and for all $\mathscr{b} \in \mathfrak{C} = \mathcal{A} \cup \mathfrak{B}$, it is either $\mathscr{b} \in \mathcal{A} \backslash \mathfrak{B}$ or $\mathscr{b} \in \mathfrak{B} \backslash \mathcal{A}$. If $\mathscr{b} \in \mathcal{A} \backslash \mathfrak{B}$ then $\mathcal{H}(\mathscr{b}) = \mathfrak{F}(\mathscr{b})$ and if $\mathscr{b} \in \mathfrak{B} \backslash \mathcal{A}$, then $\mathcal{H}(\mathscr{b}) = \mathfrak{G}(\mathscr{b})$. In both cases, $\mathcal{H}(\mathscr{b})$ is a regular subsemigroup of $\mathcal{S}$. Therefore, $(\mathcal{H}, \mathfrak{C})$ is a soft regular semigroup over $\mathcal{S}$.

Lemma 3.4: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be soft regular semigroups over $\mathcal{S}$ such that $(\mathfrak{F}, \mathcal{A}) \cap_{\mathfrak{R}} (\mathfrak{G}, \mathfrak{B})$ is non null and non empty. Then their *restricted intersection*, $(\mathfrak{F}, \mathcal{A}) \cap_{\mathfrak{R}} (\mathfrak{G}, \mathfrak{B})$ is also a soft regular semigroup over $\mathcal{S}$.

Proof: Since $(\mathfrak{F}, \mathcal{A}) \cap_{\mathfrak{R}} (\mathfrak{G}, \mathfrak{B}) = (\mathcal{H}, \mathfrak{C})$ where $\mathfrak{C} = \mathcal{A} \cap \mathfrak{B} \neq \emptyset$ and $\mathcal{H}(\mathscr{b}) = \mathfrak{F}(\mathscr{b}) \cap \mathfrak{G}(\mathscr{b})$ for all $\mathscr{b} \in \mathfrak{C}$. It is clear that $\mathcal{H}(\mathscr{b})$ is either empty or a regular subsemigroup of $\mathcal{S}$. Consequently, $(\mathcal{H}, \mathfrak{C})$ is a soft regular semigroup over $\mathcal{S}$.

Lemma 3.5: Let $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$ be two soft regular semigroups over $\mathcal{S}$. Then the *basic intersection* $(\mathfrak{F}, \mathcal{A}) \wedge (\mathfrak{G}, \mathfrak{B})$ of $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$ is also a soft regular semigroup whenever $(\mathfrak{F}, \mathcal{A}) \wedge (\mathfrak{G}, \mathfrak{B})$ is non null, where $\mathcal{H}(\mathfrak{a}, \mathscr{b}) = \mathfrak{F}(\mathfrak{a}) \cap \mathfrak{G}(\mathscr{b})$ for all $(\mathfrak{a}, \mathscr{b}) \in \mathcal{A} \times \mathfrak{B}$.

Proof: Now, $(\mathfrak{F}, \mathcal{A}) \wedge (\mathfrak{G}, \mathfrak{B}) = (\mathcal{H}, \mathcal{A} \times \mathfrak{B})$ and $\mathcal{H}(\mathfrak{a}, \mathcal{b}) = \mathfrak{F}(\mathfrak{a}) \cap \mathfrak{G}(\mathcal{b})$ for all $(\mathfrak{a}, \mathcal{b}) \in \mathcal{A} \times \mathfrak{B}$. Since $\mathfrak{F}(\mathfrak{a})$ and $\mathfrak{G}(\mathcal{b})$ are regular subsemigroups of $\mathcal{S}$, either $\mathfrak{F}(\mathfrak{a}) \cap \mathfrak{G}(\mathcal{b}) = \emptyset$ or $\mathfrak{F}(\mathfrak{a}) \cap \mathfrak{G}(\mathcal{b})$ is a regular subsemigroup of $\mathcal{S}$. As a result, $(\mathcal{H}, \mathcal{A} \times \mathfrak{B})$ is a soft regular semigroup of $\mathcal{S}$.

Lemma 3.6: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be two soft regular semigroups over $\mathcal{S}$. Then the *basic union* $(\mathfrak{F}, \mathcal{A}) \vee (\mathfrak{G}, \mathfrak{B})$ of $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ is also a soft regular semigroup over $\mathcal{S}$.

Proof: Now, $(\mathfrak{F}, \mathcal{A}) \vee (\mathfrak{G}, \mathfrak{B}) = (\mathfrak{J}, \mathcal{A} \times \mathfrak{B})$ and $\mathfrak{J}(\mathfrak{a}, \mathcal{b}) = \mathfrak{F}(\mathfrak{a}) \cup \mathfrak{G}(\mathcal{b})\ \forall\ (\mathfrak{a}, \mathcal{b}) \in \mathcal{A} \times \mathfrak{B}$. Since $\mathfrak{F}(\mathfrak{a})$ and $\mathfrak{G}(\mathcal{b})$ are regular subsemigroups of $\mathcal{S}$, it follows that $\mathfrak{F}(\mathfrak{a}) \cup \mathfrak{G}(\mathcal{b})$is a regular subsemigroup of $\mathcal{S}$. As a result, we have that $(\mathfrak{J}, \mathcal{A} \times \mathfrak{B})$ is a soft regular semigroup over $\mathcal{S}$.

Soft Inverse Semigroup:

Definition 3.7: Let $\mathcal{S}$ be a semigroup and let $(\mathfrak{F}, \mathcal{A})$ be a soft set over $\mathcal{S}$. Then $(\mathfrak{F}, \mathcal{A})$ is called a soft inverse semigroup over $\mathcal{S}$ if for every $\mathcal{b} \in \mathcal{A}$, $\mathfrak{F}(\mathcal{b})$ is an inverse subsemigroup of $\mathcal{S}$.

Example 3.8: Let $\mathcal{S} = \{\mathfrak{h}, \mathfrak{q}, \mathfrak{u}, \mathfrak{s}, \mathfrak{w}\}$ be a semigroup with the multiplication Table 4.

Table 4. Soft inverse semigroup over $\mathcal{S}$

	$\mathfrak{h}$	$\mathfrak{q}$	$\mathfrak{u}$	$\mathfrak{s}$	$\mathfrak{w}$
$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{q}$	$\mathfrak{h}$	$\mathfrak{s}$	$\mathfrak{h}$
$\mathfrak{q}$	$\mathfrak{h}$	$\mathfrak{q}$	$\mathfrak{u}$	$\mathfrak{h}$	$\mathfrak{h}$
$\mathfrak{u}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{q}$	$\mathfrak{u}$
$\mathfrak{s}$	$\mathfrak{h}$	$\mathfrak{s}$	$\mathfrak{w}$	$\mathfrak{h}$	$\mathfrak{h}$
$\mathfrak{w}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{s}$	$\mathfrak{w}$

It is clear that $\mathcal{S}$ is not an inverse semigroup. Now, let $\mathcal{A} = \{1, 2, 3\}$ and let $\mathfrak{F}(1) = \{\mathfrak{h}, \mathfrak{q}\}$, $\mathfrak{F}(2) = \{\mathfrak{h}\}$ and $\mathfrak{F}(3) = \{\mathfrak{w}\}$, then $(\mathfrak{F}, \mathcal{A})$is a soft inverse semigoup over $\mathcal{S}$.

This example clearly shows that being a soft inverse semigroup does not imply being an inverse semigroup.

Definition 3.9: $(\mathfrak{F}, \mathcal{A})$is called an *absolute soft inverse semigroup* over $\mathcal{S}$ if $\mathfrak{F}(\mathcal{b}) = \mathcal{S}\ \ \forall\ \mathcal{b} \in \mathcal{A}$.

The following results follow in like manner with those of soft regular semigroups.

Lemma 3.10: Let $(\mathfrak{F}, \mathcal{A})$and $(\mathfrak{G}, \mathfrak{B})$ be soft inverse semigroups over $\mathcal{S}$ such that $\mathcal{A} \cap \mathfrak{B} = \emptyset$. Then their *extended union*, $(\mathfrak{F}, \mathcal{A}) \cup_{\mathrm{E}\mathfrak{x}} \mathcal{F}(\mathfrak{G}, \mathfrak{B})$ is also a soft inverse semigroup over $\mathcal{S}$.

Lemma 3.11: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be soft regular semigroups over $\mathcal{S}$ such that $\mathcal{A} \cap \mathfrak{B} = \emptyset$. Then their *extended intersection*, $(\mathfrak{F}, \mathcal{A}) \cap_{\mathrm{E}\mathfrak{x}} \mathcal{F}(\mathfrak{G}, \mathfrak{B})$ is also a soft regular semigroup over $\mathcal{S}$.

Lemma 3.12: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be soft inverse semigroups over $\mathcal{S}$ such that $(\mathfrak{F}, \mathcal{A}) \cap_{\mathfrak{R}} (\mathfrak{G}, \mathfrak{B})$ is non null and non empty. Then their *restricted intersection*, $(\mathfrak{F}, \mathcal{A}) \cap_{\mathfrak{R}} (\mathfrak{G}, \mathfrak{B})$ is also a soft inverse semigroup over $\mathcal{S}$.

Lemma 3.13: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be two soft inverse semigroups over $\mathcal{S}$. Then the *basic intersection* $(\mathfrak{F}, \mathcal{A}) \wedge (\mathfrak{G}, \mathfrak{B})$ of $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ is also a soft inverse semigroup whenever $(\mathfrak{F}, \mathcal{A}) \wedge (\mathfrak{G}, \mathfrak{B})$ is non null, where $\mathcal{H}(\mathfrak{a}, \mathcal{b}) = \mathfrak{F}(\mathfrak{a}) \cap \mathfrak{G}(\mathcal{b})$ for all $(\mathfrak{a}, \mathcal{b}) \in \mathcal{A} \times \mathfrak{B}$.

Lemma 3.14: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be two soft inverse semigroups over $\mathcal{S}$. Then the *basic union* $(\mathfrak{F}, \mathcal{A}) \vee (\mathfrak{G}, \mathfrak{B})$ of $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ is also a soft inverse semigroup over $\mathcal{S}$.

The proofs of the above results follow in the same way as the proofs of lemmas 3.2 to 3.6.

Definition 3.15: Let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be two soft inverse semigroups over $\mathcal{S}$. If $\mathfrak{B} \subseteq \mathcal{A}$ and for all $\mathscr{b} \in \mathfrak{B}$, $\mathfrak{G}(\mathscr{b})$ is a subsemigroup of $\mathfrak{F}(\mathscr{b})$, then $(\mathfrak{G}, \mathfrak{B})$ is called a soft inverse subsemigroup of $(\mathfrak{F}, \mathcal{A})$.

Example 3.16: Consider example 3.7 above. Let $\mathfrak{B} = \{1, 2\}$ with $\mathfrak{G}(1) = \{\mathfrak{q}\}$ and $\mathfrak{G}(2) = \{\mathfrak{h}\}$, then it is clear that $(\mathfrak{G}, \mathfrak{B})$ is a soft inverse subsemigroup of $(\mathfrak{F}, \mathcal{A})$.

Theorem 3.17: If $(\mathfrak{F}, \mathcal{A})$ is soft inverse semigroup over $\mathcal{S}$, then $(\mathfrak{F}, \mathcal{A})$ is a soft regular semigroup.

Proof: It is Obvious.

Theorem 3.18: Let $(\mathfrak{F}, \mathcal{A})$ be a softinverse semigroup over $\mathcal{S}$ with cover. Then, $\mathcal{S}$ is an inverse semigroup.

Proof: Let $(\mathfrak{F}, \mathcal{A})$ be a soft inverse semigroup over $\mathcal{S}$. Then $\mathfrak{F}(\mathscr{b})$ is an inverse for each $\mathscr{b} \in \mathcal{A}$. Now let $\mathfrak{a} \in \mathcal{S}$. Since $\bigcup_{\mathscr{b} \in \mathcal{A}} \mathfrak{F}(\mathscr{b}) = \mathcal{S}$, there exist some $\mathscr{g} \in \mathcal{A} \ni \mathfrak{a} \in \mathfrak{F}(\mathscr{g})$. Since $\mathfrak{F}(\mathscr{g})$ is inverse, there exists a unigue $\mathfrak{w} \in \mathfrak{F}(\mathscr{g})$ such that $\mathfrak{a} = \mathfrak{a}\,\mathfrak{w}\,\mathfrak{a}$ and $\mathfrak{w} = \mathfrak{w}\,\mathfrak{a}\,\mathfrak{w}$. Since $\in \mathfrak{F}(\mathscr{g}) \subseteq \bigcup_{\mathscr{b} \in \mathcal{A}} \mathfrak{F}(\mathscr{b}) = \mathcal{S}$, $\mathcal{S}$ is inverse.

Finally, we consider the concept of homomorphism between two soft inverse semigroups, what is known as soft inverse homomorphism (or morphism).

Definition 3.19: Let $\mathcal{S}$ and $\mathcal{W}$ be semigroups and let $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ be soft inverse semigroups. Also, let $\tau: \mathcal{S} \to \mathcal{W}$ and $\varpi: \mathcal{A} \to \mathfrak{B}$ be mappings. Then (τ, ϖ) is a *soft inverse homomorphism* if the following holds:

i) τ is an epimorphism from $\mathcal{S}$ to $\mathcal{W}$
ii) ϖ is an onto mapping from $\mathcal{A}$ to $\mathfrak{B}$
iii) $\tau\big(\mathfrak{F}(\mathscr{b})\big) = \mathfrak{G}\big(\varpi(\mathscr{b})\big)$ for all $\mathscr{b} \in \mathcal{A}$.

As a result, $(\mathfrak{F}, \mathcal{A})$ is said to be *soft inverse homomorphic* to $(\mathfrak{G}, \mathfrak{B})$

If τ is an isomorphism from $\mathcal{S}$ to $\mathcal{W}$ and ϖ is one-to-one from $\mathcal{A}$ to $\mathfrak{B}$, then (τ, ϖ) becomes a *soft inverse isomorphism*. When that happens, we write $(\mathfrak{F}, \mathcal{A}) \cong (\mathfrak{G}, \mathfrak{B})$ and read it as "$(\mathfrak{F}, \mathcal{A})$ is softinverse isomorphic to $(\mathfrak{G}, \mathfrak{B})$."

The following properties of soft inverse homomorphism (or morphism) follow from the homomorphism between inverse semigroups and the definition of soft inverse homomorphism:

Lemma 3.20: Let $\mathcal{S}$ and $\mathcal{W}$ be two inverse semigroups and let (τ, ϖ) be a soft inverse homomorphism from $(\mathfrak{F}, \mathcal{A})$ to $(\mathfrak{G}, \mathfrak{B})$, where $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ are two soft inverse semigroups. If $(\mathfrak{F}, \mathcal{A})$ is idempotent over $\mathcal{S}$, then $(\mathfrak{G}, \mathfrak{B})$ is idempotent over $\mathcal{W}$.

Proof: $(\mathfrak{F}, \mathcal{A})$ is idempotent over $\mathcal{S}$ means $\mathfrak{F}(\mathscr{b})$ is an idempotent subsemigroup of $\mathcal{S}$. Now, since (τ, ϖ) is a soft inverse homomorphism, we have that for any $\mathfrak{d} \in \mathfrak{B}$ there is $\mathscr{b} \in \mathcal{A}$ such that $\varpi(\mathscr{b}) = \mathfrak{d}$. Hence, we must have that

$$\mathfrak{G}(\mathfrak{d}) = \mathfrak{G}\big(\varpi(\mathscr{b})\big) = \tau\big(\mathfrak{F}(\mathscr{b})\big).$$

Since $\mathfrak{F}(\mathscr{b})$ is idempotent over $\mathcal{S}$, by implication $\tau\big(\mathfrak{F}(\mathscr{b})\big)$ is idempotent over $\mathcal{W}$. Therefore, $\mathfrak{G}(\mathfrak{d})$ is idempotent and so, $(\mathfrak{G}, \mathfrak{B})$ is idempotent over $\mathcal{W}$.

Lemma 3.21: Let $\mathcal{S}$ and $\mathcal{W}$ be two inverse semigroups and let (τ, ϖ) be a soft inverse homomorphism from $(\mathfrak{F}, \mathcal{A})$ to $(\mathfrak{G}, \mathfrak{B})$, where $(\mathfrak{F}, \mathcal{A})$ and $(\mathfrak{G}, \mathfrak{B})$ are two soft inverse semigroups. If $(\mathfrak{F}, \mathcal{A})$ is regular over $\mathcal{S}$, then $(\mathfrak{G}, \mathfrak{B})$ is regular over $\mathcal{W}$.

Proof: It is clear, since $(\mathfrak{F}, \mathcal{A})$ is soft inverse isomorphic to $(\mathfrak{G}, \mathfrak{B})$ [25,26].

4 Conclusion

Ali et al. [4] introduced soft regular semigroups and characterized them in terms of soft right and soft left ideals. In this paper we have presented more properties of soft regular semigroups and have also introduced soft inverse semigroups. We have presented some its properties and have also studied soft inverse morphism.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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Competing interests

Authors have declared that no competing interests exist.

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