Perfect Dominating Set of An Interval-Valued Fuzzy Graphs

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Authors' contributions

This work was carried out in collaboration between both authors. Author FMAA designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author MMQS managed the analyses of the study. Author MMQS managed the literature searches. Both authors read and approved the final manuscript.

Original Research Article

Abstract

Aims/ Objectives: Perfect domination is very much useful in network theory, Electrical stations and several fields of mathematics. In This paper, perfect domination in an interval-valued fuzzy graphs is defined and studied. Some bounds on perfect domination number $\gamma_p(G)$ are provided for several interval-valued fuzzy graphs, such as complete, wheel and star,.. etc. Furthermore, the relationship of $\gamma_p(G)$: with some other known parameters in interval-valued fuzzy graphs investigated with some suitable examples.

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1   Introduction

Zadeh in (1975)[1] introduced the idea of interval-valued fuzzy sets as an extension of fuzzy sets, which gives a more precise tool to model uncertainty in real-life situations. Interval-valued fuzzy sets have been widely used in many areas of science and engineering. Rosenfeld introduced another detailed definition for each fuzzy vertex, fuzzy edges and several fuzzy analogs of graph theoretic concepts such as paths, cycles, connectedness etc [2]. Akram and Dudek [3] studied several properties and operations on interval-valued fuzzy graphs. The concept of domination in fuzzy graphs was investigated by A. SomaSundaram and S. SomaSundaram [4] further A. SomaSundaram presented the concepts of independent domination, total domination, connected domination of fuzzy graphs [5]. In (2013) Revathi introduced the concept of perfect domination in fuzzy graphs [6]. The concepts of domination in interval valued fuzzy graphs was investigated by Pradip Debnath [7], Because of the Large area for dominating applications in the real-life, such as computer networks and the internets network. Sarala in (2016) [8] introduced the concepts of strong (weak) domination in interval-valued fuzzy graphs. We introduce and study the concept of perfect domination in interval-valued fuzzy graphs. In this paper we introduce and investigate the concept of perfect domination number in interval-valued fuzzy graphs, we obtain many results related to this concept. The relationship between this concept and the others in interval-valued fuzzy graph will be given.

2   Preliminaries

We review in this section some basic definitions related to interval-valued fuzzy graphs and domination in interval-valued fuzzy graph.

A subset A of a vertex set V is called interval valued fuzzy set and it is denoted by $A = \{u, [\mu_1(u), \mu_2(u)] : u \in V\}$, where the function $\mu_1 : V \to [0,1]$ and $\mu_2 : V \to [0,1]$, such that $\mu_1(u) \le \mu_2(u)$ for $u \in V$. If $G^* = (V, E)$ is a crisp graph, then by an interval valued fuzzy relation $\rho = (\rho_1, \rho_2)$ on V, we mean an interval valued fuzzy on E, such that

$$\rho_1(u, v) \le \mu_1(u) \wedge \mu_1(v)$$

and

$$\rho_2(u, v) \le \mu_2(u) \wedge \mu_2(v)$$

for all $(u, v) \in E$ and denoted by $B = \{(u, v), [\rho_1(u, v), \rho_2(u, v)] : (u, v) \in E\}$, where the function $\rho_1 : E \to [0,1]$ and $\rho_2 : E \to [0,1]$, such that $\rho_1(u, v) \le \rho_2(u, v)$ for $(u, v) \in E$. An interval-valued fuzzy graph of the graph $G^* = (V, E)$ is a pair $G = (A, B)$ where
(1) $A = [\mu_1, \mu_2]$ is an interval-valued fuzzy set on V.
(2) $B = [\rho_1, \rho_2]$ is an interval-valued fuzzy relation on V, such that

$$\rho_1(u, v) \le min\{\mu_1(u), \mu_1(v)\}$$

and

$$\rho_2(u, v) \le min\{\mu_2(u), \mu_2(v)\}$$

for all $(u, v) \in E$.

Let $G = (A, B)$ be an interval-valued fuzzy graph. Then the cardinality of G is defined to be

$$|G| = \sum_{v_i \in V} \frac{1 + \mu_2(v_i) - \mu_1(v_i)}{2} + \sum_{(v_i, v_j) \in E} \frac{1 + \rho_2(v_i, v_j) - \rho_1(v_i, v_j)}{2}.$$

The vertex cardinality of G is defined by

$$|V| = \sum_{v_i \in V} \frac{1 + \mu_2(v_i) - \mu_1(v_i)}{2}.$$

For all $v_i \in V$ is called the order of an interval-valued fuzzy graph is denoted by $P(G)$. The edge cardinality of an interval-valued fuzzy graph G is defined by

$$|E| = \sum_{(v_i, v_j) \in E} \frac{1 + \rho_2(v_i, v_j) - \rho_1(v_i, v_j)}{2}.$$

For all $(v_i, v_j) \in E$ is called the size of an intreval-valued fuzzy graph is denoted by $q(G)$. An edge $e = (x, y)$ of an interval-valued fuzzy graph is called effective edge if $\rho_1(x, y) = min\{\mu_1(x), \mu_1(y)\}$ and $\rho_2(x, y) = min\{\mu_2(x), \mu_2(y)\}$. The degree of a vertex can be generalized in different ways for an interval-valud fuzzy graph $G = (A, B)$ of $G^* = (V, E)$. The effective degree of a vertex v in an interval-valued fuzzy graph, $G = (A, B)$ is defined to be sum of the weights of the effective edges incident at v and it is denoted by $d_E(v)$. The minimum effictive edges degree of G is $\delta_E(G) = min\{d_E(v)|v \in V\}$. The maximum effective degree of G is $\Delta_E(G) = max\{d_E(v)|v \in V\}$. Two vertices u and v are said to be neighbors in interval-valued fuzzy graph $G = (A, B)$ of $G^* = (V, E)$ if $\rho_1(u, v) = \mu_1(u) \wedge \mu_1(v)$, and $\rho_2(u, v) = \mu_2(u) \wedge \mu_2(v)$. A vertex subset $N(v) = \{u \in V : v \text{ adjacent to } u\}$ is called the open neighborhood set of a vertex v and $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood set of v. The neighborhood degree of a vertex v in an interval-valued fuzzy graph, $G = (A, B)$ is defined to be sum of the weights of the vertices adjacent to v, and it is denoted by $d_N(v)$, that is mean that $d_N(v) = |N(v)|$. The minimum neighborhood degree of G is $\delta_N(G) = min\{d_N(v)|v \in V\}$. The maximum neighborhood degree of G is $\Delta_N(G) = max\{d_N(v)|v \in V\}$. The neighborhood degree of a vertex v in an interval-vaiued fuzzy graph $G = (A, B)$ is defined to be the sum of the weights of the vertices which adjacent to v and is denoted by $d_s(v)$. An interval-valued fuzzy graph, $G = (A, B)$ is said to be complete interval-valued fuzzy graph if $\rho_1(v_i, v_j) = min\{\mu_1(v_i), \mu_1(v_j)\}$ and $\rho_2(v_i, v_j) = min\{\mu_2(v_i), \mu_2(v_j)\}$, for all $v_i, v_j \in V$ and is denoted by K_μ. The complement of an interval-valued fuzzy graph, $G = (A, B)$ is an interval-valued fuzzy graph, $\overline{G} = (\overline{V}, \overline{E})$ where

(i) $\overline{V} = V$;
(ii) $\overline{\mu_1} = \mu_1$; $\overline{\mu_2} = \mu_2$ for all vertices;
(iii) $\overline{\rho_1}(u, v) = min\{\mu_1(u), \mu_1(v)\} - \rho_1(u, v)$ and $\overline{\rho_2}(u, v) = min\{\mu_2(u), \mu_2(v)\} - \rho_2(u, v)$ for all $u, v \in E$.

An interval-valued fuzzy graph $G = (A, B)$ is said to bipartite if the vertex set V of G can be bartitioned into two non empty sub sets V_1 and V_2 such that $V_1 \cap V_2 = \phi$.
A bipartite interval-valued fuzzy graph $G = (A, B)$ is said to be complete bipartite interval-valued fuzzy graph if $\rho_1(v_i, v_j) = min\{\mu_1(v_i), \mu_1(v_j)\}$ and $\rho_2(v_i, v_j) = min\{\mu_2(v_i), \mu_2(v_j)\}$ for all $v_i \in V_1$ and $v_j \in V_2$. Its denoted by $K_{m,n}$, where $|V_1| = m, |V_2| = n$. A vertex $u \in V$ of an interval-valued fuzzy graph, $G = (A, B)$ is said to be an isolated vertex if $\rho_1(v, u) = 0$ and $\rho_2(v, u) = 0$ for all $v \in V$. That is $N(u) = \phi$. An interval-valued fuzzy graph, $G = (A, B)$ is said to be strong interval-valued fuzzy graph, if $\rho_1(u, v) = min(\mu_1(u), \mu_1(v))$, and $\rho_2(u, v) = min(\mu_2(u), \mu_2(v))$ for all $(u, v) \in E$. An interval-valued fuzzy graph, $H = (X, Y)$, is said to be an interval-valued fuzzy subgraph of $G = (A, B)$, if $X \subseteq A$, and $Y \subseteq B$. That is $\mu_1^\backslash \le \mu_1$, $\mu_2^\backslash \le \mu_2$ and $\rho_1^\backslash \le \rho_1$; $\rho_2^\backslash \le \rho_2$.

Let $G = (A, B)$ be an interval valued fuzzy graph and let $S \in V(G)$, then a vertex sub set S of G is said to be independent set if $\rho_1(u, v) < \mu_1(u) \wedge \mu_1(v)$, and $\rho_2(u, v) < \mu_2(u) \wedge \mu_2(v)$ or $\rho_1(u, v) = 0, \rho_2(u, v) = 0$ for all $u, v \in S$. An independent vertex sub set S of an interval valued fuzzy graph $G = (A, B)$ is said to be maximal independent set if $S \cup \{u\}$ is not independent $\forall u \in V - S$. The maximum fuzzy cardinality amony all maximal independent set in interval valued fuzzy graph, G is called the independence number of G and is denoted by $\beta_0(G)$. Let $G = (A, B)$ be an interval valued fuzzy graph and $e = (u, v) \in E(G)$, if e is effective edge. Then we say that e and u, e and v are covering each others. A vertex subset S of $V(G)$ in an in interval valued fuzzy graph is said to be vertex covering set of G if for each edge e in G there exists a vertex u in S such that u covers e. A vertex covering set of an interval valued fuzzy graph, $G = (A, B)$ is said to be minimal vertex covering set if for every $u \in S$, $S - \{u\}$ is not vertex covering set. The minimum fuzzy

cardinality amony all minimal vertex covering set of an in interval valued fuzzy graph, G is called the vertex coverage number and is denoted by $\alpha_0(G)$. Let $G = (A, B)$ be an interval-valued fuzzy graph and let $u, v \in V(G)$. Then we say that u dominates v or v dominates u if (u, v) is a effective edge, i.e $\rho_1(u, v) = min(\mu_1(u), \mu_1(v))$ and $\rho_2(u, v) = min(\mu_2(u), \mu_2(v))$. A vertex sub set $(D \subseteq V)$ of $V(G)$ is called dominating set in an interval-valued fuzzy graph G, if for every $v \in V - D$ there exists $u \in D$, such that (u, v) is effective edge. A dominating set D of an interval valued fuzzy graph G is called minimal dominating set if $D - \{u\}$ is not dominating set for every $u \in D$. A minimal dominating set D, with $|D| = \gamma(G)$ is denoted by $\gamma - set$.

3 Perfect Dominating Set Of an Interval-valued Fuzzy Graph

Definition 3.1. Let $G = (A, B)$ be an interval-valued fuzzy graph and $x, y \in V$, the vertex x dominates the vertex y in G if $\rho_1(x, y) = min\{\mu_1(x), \mu_1(y)\}$ and $\rho_2(x, y) = min\{\mu_2(x), \mu_2(y)\}$, for all $x, y \in V$.

Definition 3.2. A dominating set D of an interval-valued fuzzy graph, G is called the perfect dominating set of G if for each vertex x not in D, x is dominates by exactly one vertex of D and is doneted by D_p.

Definition 3.3. A perfect dominating set D_p of interval-valued fuzzy graph, G is called minimal perfect dominating set, if for $u \in D_p$, $D_p - \{u\}$ is not perfect dominating set of G .

Definition 3.4. The minimum fuzzy cardinality among all minimal perfect dominating sets of an interval-valued fuzzy graph, G is called the perfect domination number of G and denoted by $\gamma_p(G)$ or γ_p.

Remark 3.1. A perfect dominating set of interval-valued fuzzy graph, G with smallest fuzzy cardinality, i.e $|D_p| = \gamma_p(G)$, is called the minimum perfect dominating set and denoted by $\gamma_p - set$.

Remark 3.2. In interval-valued fuzzy graph G, for $(x, y) \in V$, if x dominates y then y dominates x and as such domination is a symetric relation. But in the perfect dominating need not it is true, i.e perfectionism domination is not symetric relation on V.

The following Example given the γ_p of an interval-valued fuzzy graph.

Example 3.1. *For the interval-valued fuzzy graph G shown in Figure 1, such that every edge in G is strong edge.*
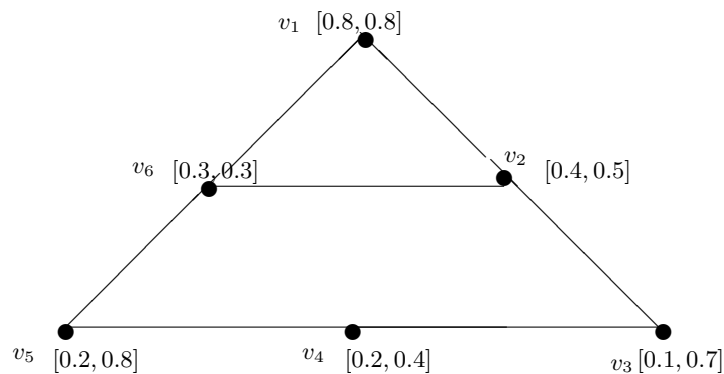

Fig. 1

From Fig. 1, we see that $D_{p1} = \{v_1, v_4\}, D_{p2} = \{v_1, v_2, v_6, v_5\}$ and $D_{p3} = \{v_1, v_2, v_6, v_3\}$ are minimal perfect dominating sets of G. So, $\gamma_p(G) = min\{|D_{p1}|, |D_{p2}|, |D_{p3}|\} = min\{1.1, 2.45, 2.45\} = 1.1$.

Theorem 3.2. *Every perfect dominating set of an interval-valued fuzzy graph G is a dominating set of an interval-valued fuzzy graph G. But the converse is need not true.*

Proof. Let $G = (A, B)$ be an interval-valued fuzzy graph, by definition of the perfect dominating set of G, if for each vertex u is not in D_p. Then u dominated by exactly one vertex of D_p. Hence it is clear that every vertex u is not in D_p, $u \in V - D_p$ and u a strongly to exactly one vertex in D_p. Hence D_p is a dominating set of G. $\qquad \square$

To show that the converse of the above Theorem, need not be true, from Figure (3.1), we have $D = \{v_2, v_4\}$ is dominating set, but it is not perfect dominating set of G.

Corollary 3.3. *For any interval-valued fuzzy graph G.*

$$\gamma(G) \leq \gamma_p(G).$$

Proof. By above Theorem, since every perfect dominating set of interval-valued fuzzy graph, G is dominating set of G. Hence $\gamma(G) \leq \gamma_p(G)$. $\qquad \square$

In the following we give γ_p for The complete interval-valued fuzzy graph.

Theorem 3.4. *For any complete interval-valued fuzzy graph $G = k_\mu$.*

$$\gamma_p(k_\mu) = min\{|u|; \forall u \in V(G)\}.$$

Proof. Let G be complete interval-valued fuzzy graph. Then all edges in σ are strong edge and each vertex in k_μ dominates all the other vertices in σ. Hence a perfect dominating set contain exactly one vertex say v in k_μ, where v has minimum membership value in k_μ. Thus

$$\gamma_p(k_\mu) = min\{|u|; \forall u \in V(G)\}.$$

$\qquad \square$

Theorem 3.5. *Let G be any complete interval-valued fuzzy graph and let $\overline{G}$ be the complementary of G. Then perfect dominating set in $\overline{G}$ is not exists.*

Proof. Let G be any complete interval-valued fuzzy graph and let $\overline{G}$ be complement of G. Then all edges of G are a strong neighbors and every vertex in G dominates to all vertices of G, by the bove Theorem perfect dominating is exists. Since $\overline{G}$ be the comlementary of G. Hence all edges in $\overline{G}$ equal zero. Then $\overline{G}$ is null interval-valued fuzzy graph . Hence $\overline{\gamma}_p = 0$. $\qquad \square$

Corollary 3.6. *For any complete interval-valued fuzzy graph G.*

$$\gamma(k_\mu) = \gamma_p(k_\mu).$$

Corollary 3.7. *For any complete interval-valued fuzzy graph G.*

$$\gamma(\overline{k_\mu}) \neq \gamma_p(\overline{k_\mu}).$$

Example 3.8. *For the interval-valued fuzzy graph G, shown in Figure 2, such that all edges of G are effective.*

Fig 2

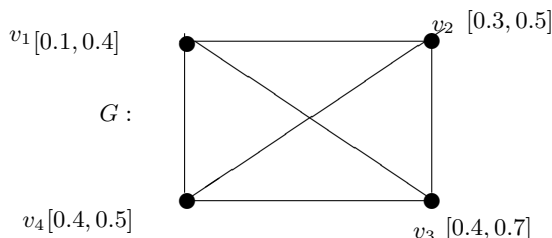*From the Fig. (2), we see that all vertices of G are minimal perfect dominating set. (i.e) a vertex subset of G, $D_1 = \{v_1\}, D_2 = \{v_2\}, D_3 = \{v_3\}$ and $D_4 = \{v_4\}$ are minimal perfect dominating sets. Then $\gamma(G) = \gamma_p(G) = min\{|D_1|, |D_2|, |D_3|, |D_4|\} = min\{0.65, 0.6, 0.75, 0.55\} = 0.55$*

The following Fig. 3, give the complement of interval-valued fuzzy graph G.

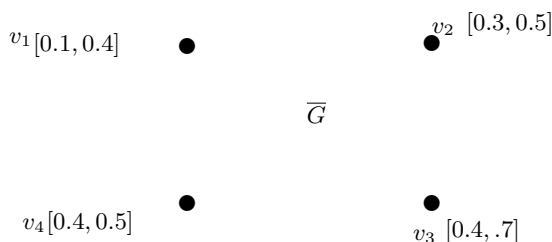**Example 3.9.** *For the interval-valued fuzzy graph G, shown in Figure 3.*

Fig 3

From the Fig. (3), we see that dominating set of $\overline{G} = \{v_1, v_2, v_3, v_4\} = V$. Then $\gamma(\overline{G}) = 2.55$, and the perfect domination number $\gamma_p(\overline{G}) = 0$. Hence

$$\gamma(\overline{G}) \neq \gamma_p(\overline{G}).$$

Remark 3.3. For any interval-valued fuzzy graph, G if $V(G)$ is an independent. Then

$$\gamma_p(G) = 0, \ but \gamma(G) = p.$$

In the following we give γ_p for The complete bipartite interval-valued fuzzy graph G.

Theorem 3.10. *For any complete bipartite interval-valued fuzzy graph, $G = k_{\mu_1, \mu_2}$*

$$\gamma_p(G) = min\{|u| : u \in V_1\} + min\{|v| : v \in V_2\}.$$

Proof. Let $G = (A, B)$ be complete bipartite $IVFG$, then all edges in G are strong arcs and each vertex in V_1 dominates all vertices in V_2 and each vertex in V_2 dominates all vertices in V_1. Hence a perfect dominating set of G contain exactly two vertices u, v where $u \in V_1$ and $v \in V_2$, where u has the minimum membership value in V_1 and v has the minimum membership value of V_2. Therefour,

$$\gamma_p(G) = min\{|u| : u \in V_1\} + min\{|v| : v \in V_2\}.$$

$\square$

Corollary 3.11. *Let $G = k_{\mu_1,\mu_2}$ be a complete bipartite IVFG. Then*

$$\gamma(G) = \gamma_p(G) = \gamma_p(\overline{G}).$$

In the following we give γ_p for The star interval-valued fuzzy graph .

Theorem 3.12. *Let G be a strong star interval-valued fuzzy graph. Then $\gamma_p(G) = |u|$, such that u is a root vertex.*

Proof. Let G be a strong star interval-valued fuzzy graph, then the vertex set of G are $\{u, v_i\}$, where u the root vertex of G and $v_i = \{v_1, v_2, ..., v_{n-1}\}$, such that u dominates all vertices v_i, for $i = \{1, 2, ..., n-1\}$. Hence perfect dominating set contains $\{u\}$. Therefour, the perfect domination number $\gamma_p(G) = |u| : \ u$ is a root vertex, where,

$$\gamma_p(G) = |u| = \frac{1 + \mu_2(u) - \mu_1(u)}{2}.$$

<div align="right">□</div>

Corollary 3.13. *For any strong star interval-valued fuzzy graph G,*

$$\gamma(G) = \gamma_p(G).$$

Example 3.14. *For the interval-valued fuzzy graph G, shown in Fig. 4, such that all edges of G are strong*

$v_4 [0.2, 0.9]$

$v_1 [0.1, 0.2]$ $v_2 \ [0.1, 0.5]$ $v_5 \ [0.3, 0.6]$
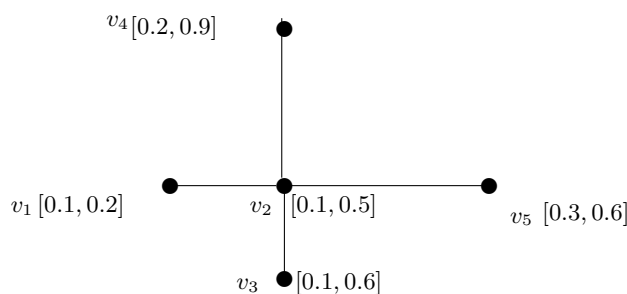
$v_3 \ [0.1, 0.6]$

Fig. 4.

From the Fig (4), we have $\gamma(G) = \{v_2\} = \gamma_p(G) = 0.65$.

Corollary 3.15. *Let G be a strong star interval-valued fuzzy graph. Then*

$$\gamma_p(G) \neq \gamma_p(\overline{G}).$$

Theorem 3.16. *Let G be a strong wheel interval-valued fuzzy graph. Then $\gamma_p(G) = |u| : u$ is a root vertex.*

Proof. Let G be a strong wheel interval-valued fuzzy graph. Then the vertex set of $V(G)$ aqual to $\{u, v_i\}$, $i = 1, 2, ..., n-1$, where u the root vertex of G, and $v_i = \{v_1, v_2, ..., v_{n-1}\}$. Then a root vertex u dominates v_i, for $i = \{1, 2, ..., n\}$. Hence perfect dominating set contains only the a root vertex u. Therefore the perfect domination number $\gamma_p(G) = |u|$, u is a root vertex, where

$$\gamma_p(G) = |u| = \frac{1 + \mu_2(u) - \mu_1(u)}{2}.$$

<div align="right">□</div>

Corollary 3.17. *For wheel interval-valued fuzzy graph G,*

$$\gamma(G) = \gamma_p(G).$$

Corollary 3.18. *Let G be a strong wheel interval-valued fuzzy graph. Then*

$$\gamma_p(G) \le \gamma_p(\overline{G}).$$

Theorem 3.19. *A perfect dominating set D_p of an interval-valued fuzzy graph G, is minimal perfect dominating set if and only if one of the following conditions hold:*

(i) $N(v) \cap D_p = \phi$.

(ii) There is a vertex $u \in V - D_p$, such that $N(u) \cap D_p = \{v\}$.

Proof. Let G be interval-valued fuzzy graph, let D_p be $\gamma_p -$ set of G and let $v \in D_p$. Then $D_p - \{v\}$ is not perfect dominating set of G and there exists a vertex $u \in V - \{D_p - \{v\}\}$, such that u is not dominated by any vertex of $D_p - \{v\}$, if $u = v$, then $N(v) \cap D_p = \phi$, if $u \ne v$, then $N(u) \cap D_p = \{v\}$. **Conversely:** suppose that D_p is a perfect dominating set and for each a vertex $v \in D_p$ one of two conditions holds. Suppose that D_p is not minimal perfect dominating set, then there exsits a vertex $v \in D_p$; $D_p - \{v\}$ is a perfect dominating set. Hence v is adjacent to at least one vertex in $D_p - \{v\}$, then the condition one dose not hold. If $D_p - \{v\}$ is perfect dominating set then every vertex in $V - D_p$ is adjacent to at least one vertex in $D_p - \{v\}$, then the condition two dose not hold, which a contradiction. So D_p is minimal perfect dominating set of G. □

Theorem 3.20. *Let G be an interval-valued fuzzy graph without isolated vertices, $n \ge 5$ and D_p be a minimal perfect dominating set of G. Then $V - D_p$ need not be perfect dominating set of G.*

Proof. Let G be an interval-valued fuzzy graph without isolated vertices, $n \ge 5$ and D_p be a minimal perfect dominating set of G. Suppose that $V - D_p$ is a perfect dominating set of G. Then by the definition of Perfect dominating set, for each vertex $u \notin V - D_p$, u is dominates by exactly one vertex of $V - D_p$. Hence there is vertex $v \in V - D_p$, such that v does not adjacent to any vertex in D_p or v dominates at least one vertex in D_p. Therfour D_p is not minimal perfect dominating set of G, which is a contradiction. Thus $V - D_p$ need not perfect dominating set of G. □

In the following Example we show that the results above Theorem.

Example 3.21. *For the interval-valued fuzzy graph G given in Fig. 5, such that all edges in G are strong.*
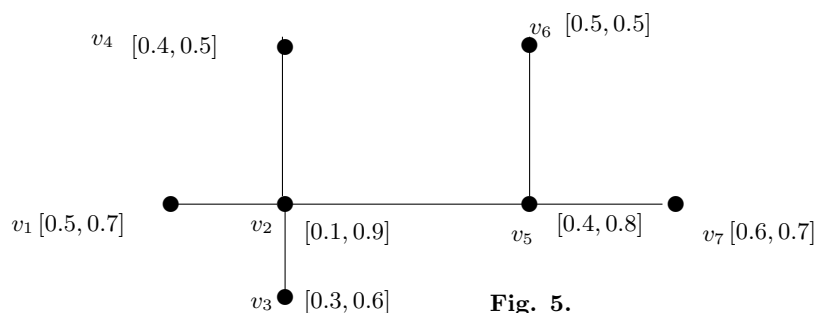

Fig. 5.

A vertex subset $D_p = \{v_2, v_5\}$, is minimal perfect dominating set. Then $V - D_p = \{v_1, v_3, v_4, v_6, v_7\}$ is not perfect dominating set of G.

Corollary 3.22. *Let G be any interval-valued fuzzy graph and let D_p be perfect dominating set of G. If the number of vertices in V is twice of the number of vertices in D_p. Then $V - D_p$ is perfect dominating set of G.*

Theorem 3.23. *Let G be interval-valued fuzzy graph without vertices, if $V(G)$ is even. Then $\gamma_p(G) \leq \frac{p}{2}$.*

Proof. Let G be interval-valued fuzzy graph has two disjoint perfect dominating sets $\gamma_{p_1}(G)$ and $\gamma_{p_2}(G)$, where the vertices of G is even, if $\gamma_{p_1}(G) = \gamma_{p_2}(G)$. Then

$$\gamma_{p_1}(G) + \gamma_{p_2}(G) \leq p.............(1).$$

If $\gamma_{p_1}(G) < \gamma_{p_2}(G)$, or $\gamma_{p_1}(G) < \gamma_{p_2}(G)$. Then

$$\gamma_{p_1}(G) + \gamma_{p_2}(G) < p..............(2).$$

From (1) and (2). Therefore,

$$\gamma_p(G) \leq \frac{p}{2}.$$

$\square$

Remark 3.4. Let G be a cycle interval-valued fuzzy graph contain odd vertices with C_{2n+1}, $n \geq 2$. Then $\gamma_p(G) \leq \frac{p}{2}$ need not always is true. .

In the following example we show that If $V(G)$ is odd of a cycle, the results above is not correct.

Example 3.24. *For interval-valued fuzzy graph given in Fig. (6), such that all edges of G are strong.*
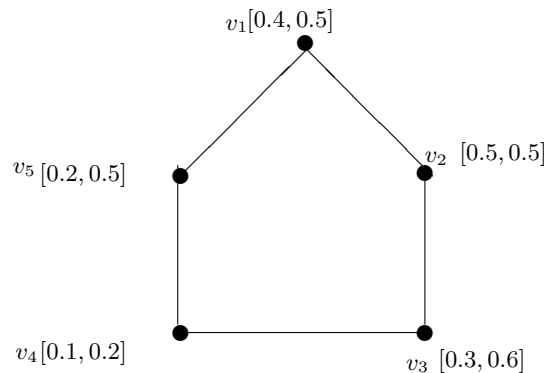

Fig. 6.

A vertex sub set $D_{p_1} = \{v_1, v_2, v_3\}$, $D_{p_2} = \{v_2, v_3, v_4\}$, $D_{p_3} = \{v_3, v_4, v_5\}$, $D_{p_4} = \{v_2, v_4, v_5\}$ and $D_{p_5} = \{v_1, v_2, v_5\}$, are minimal perfect dominating set of G.
Perfect dominatino number

$$\gamma_p(G) = min\{|D_{p_1}|, |D_{p_2}|, |D_{p_3}|, |D_{p_4}|, |D_{p_5}|\} = \{1.7, 1.7, 1.8, 1.75, 1.7\} = 1.7,$$

$p = |V| = 2.9$ and $\frac{p}{2} = \frac{2.9}{2} = 1.45$. Hence $\gamma_p(G) \geq \frac{p}{2}$.

Corollary 3.25. *Let G be interval-valued fuzzy graph such that both G and $\overline{G}$ have no isolated vertices then $\gamma_p(G) + \overline{\gamma}_p(G) \leq p$, where $\overline{\gamma}_p(G)$ the perfect domination number of $\overline{G}$ further equality holds if and only if $\gamma_p(G) = \overline{\gamma}_p(G) \leq \frac{p}{2}$.*

Example 3.26. *For interval-valued fuzzy graph given in Fig. (7), we see that, $\gamma_p(G) = \overline{\gamma}_p(G) = 1,\quad p = 2$, so that $\gamma_p(G) + \overline{\gamma}_p(G) = 2$. .*

$v_1 [0.1, 0.1]$ v_2 $[0.3, 0.3]$
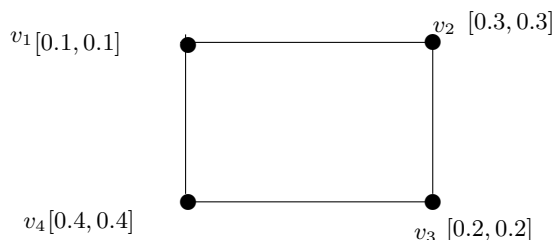
$v_4 [0.4, 0.4]$ v_3 $[0.2, 0.2]$

Fig. 7.

Remark 3.5. For any interval-valued fuzzy graph G. Then

$(1) \Delta_E(G) \geq \Delta_N(G)$ or $\Delta_E(G) \leq \Delta_N(G)$.

$(2) \Delta_E(G) = \Delta_N(G)$, if $\mu_1(x) = \mu_2(x)$ for $x \in V$.

Theorem 3.27. *For any interval-valued fuzzy graph G without isolated vertex.*

$$\gamma_p(G) \leq p - \Delta_N(G) + 1.$$

Proof. Let G be interval-valued fuzzy graph, with $\gamma_p - set$ and let v be a vertex of G with $\Delta_E(v) = d_E(v)$ and $\Delta_N(v) = d_N(v)$. Then by above Remark, either $\gamma_p \leq p - \Delta_N$ or $\gamma_p \geq p - \Delta_N$. So, we have two cases:

Case1: If $\gamma_p \leq p - \Delta_N$. Then the proof is trivial.

Case2: If $\gamma_p \geq p - \Delta_N$. Clear $\gamma_p - 1 \leq p - \Delta_N$. Hence

$$\gamma_p(G) \leq p - \Delta_N(G) + 1.$$

$\square$

Example 3.28. *From the Fig. (6), We see that $p = 2.9$, $\gamma_p(G) = 1.7$ and $\Delta_N = 1.3$. Thus $\gamma_p(G) \geq p - \Delta_N$. Hence $\gamma_p(G) \leq p - \Delta_N(G) + 1$.*

Theorem 3.29. *For any interval-valued fuzzy graph G, with at least one minimal perfect dominating set. Then*

$$\gamma(G) \leq \frac{p + \gamma_p(G)}{3}.$$

Proof. Let $G = (A, B)$ be an interval-valued fuzzy graph, with minimal perfect dominating set. Since $\gamma(G) \leq \gamma_p(G)$ and $\gamma(G) \leq \frac{p}{2}$. Hance

$$\gamma(G) \leq \frac{p + \gamma_p(G)}{3}.$$

$\square$

Example 3.30. *For interval-valued fuzzy graph G, given in Fig. (8), such that all edges of G are strong.*

$$v_1 \ (0.1, 0.5)$$

$$v_5 \ (0.2, 0.4)$$

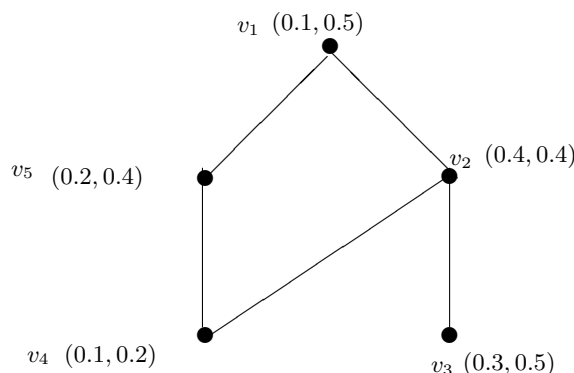$$v_2 \ (0.4, 0.4)$$

$$v_4 \ (0.1, 0.2)$$

$$v_3 \ (0.3, 0.5)$$

Fig. 8.

A vertex sub sets $D_1 = \{v_1, v_2\}, D_2 = \{(v_3, v_5)\}, D_3 = \{v_2, v_4\}, and D_4 = \{v_2, v_5\}$ are minimal dominating set of G. Then a domination number $\gamma(G) = min\{D_1, D_2, D_3, D_4\} = min\{1.2, 1.2, 1.05, 1.1\} = 1.05$. The vertex sub set $D_3 = \{v_2, v_4\}$ is a perfect dominating set of G. Then a perfect domination number of G $\gamma_p = 1.05$, and $p = 2.95$. Now $\frac{p + \gamma_p}{3} = \frac{2.95 + 1.05}{3} = 1.333$. Then $1.05 \leq 1.333$. Hence

$$\gamma(G) \leq \frac{p + \gamma_p(G)}{3}.$$

4 Conclusions

In this artical the concept of Perfect domination number which denoted by $\gamma_p(G)$ of an interval-valued fuzzy graph introduced and studied. Uper and lower bounds of $\gamma_p(G)$ in interval-valued fuzzy graph G are obtained. The relationship of $\gamma_p(G)$ and some other known parameters in interval-valued fuzzy graph G studied. Some suitable examples are given.

Competing Interests

Authors have declared that no competing interests exist.

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