Journal of Advances in Mathematics and Computer Science

24(6): 1-17, 2017; Article no.JAMCS.36489

Previously known as British Journal of Mathematics & Computer Science

ISSN: 2231-0851

Convergence of Differential Transform Method for Ordinary Differential Equations

A. S. Oke[1*]

[1] *Department of Mathematical Sciences, Faculty of Science, Adekunle Ajasin University, Akungba Akoko, Ondo State, Nigeria.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JAMCS/2017/36489

Editor(s):

(1) Lei Zhang, Winston-Salem State University, North Carolina, USA.

(2) Kai-Long Hsiao, Taiwan Shoufu University, Taiwan.

Reviewers:

(1) Sherin Zafar, Manav Rachna International University Faridabad, India.

(2) Jagdish Prakash, University of Botswana, Botswana.

(3) Alok Kumar Pandey, Roorkee Institute of Technology, India.

Complete Peer review History: http://www.sciencedomain.org/review-history/21322

Received: 30th August 2017

Accepted: 17th September 2017

Published: 10th October 2017

Original Research Article

Abstract

Differential transform method (DTM) as a method for approximating solutions to differential equations have many theorems that are often used without recourse to their proofs. In this paper, attempts are made to compile these proofs. This paper also proceeds to establish the convergence of the DTM for ordinary differential equations. This paper establishes that if the solution of an ordinary differential equation can be written as Taylors' expansion, then the solution can be obtained using the DTM. This is also demonstrated with some numerical examples.

Keywords: Differential transform; inverse differential transform; homogeneous ODE; nonhomogeneous ODE.

2010 Mathematics Subject Classification: 34L99.

**Corresponding author: E-mail: abayomi.oke@aaua.edu.ng, okeabayomisamuel@gmail.com*

1 Introduction

One of the main focus of research in recent times is on the methods for solving nonlinear ordinary differential equations since most partial differential equations can be transformed to some nonlinear ordinary differential equations by means of some rescaling terms. Perturbation Methods, Adomian Decomposition Method (ADM) ([1, 2, 3, 4, 5]), Variational Iterative Method (VIM) ([6, 7, 8]), Differential Transform Method (DTM) [9, 10, 11, 12, 13], Homotopy Perturbation Method [14] and Homotopy Analysis Method (HAM) are some of the widely used methods for solving nonlinear problems.

The Perturbation Methods, ADM, DTM and VIM are very effective in handling weakly nonlinear problems while HAM can handle weakly as well as strongly nonlinear problems.

Despite the wide applicability of the Homotopy Analysis Method, it requires the solution of some differential equations and some quadratures. When the problem involved is large, the quadratures become too cumbersome and uneasy to handle but the DTM has an advantage over this setback. The DTM reduces the problem to a set of recursive equations that can easily be handled recursively. DTM has been applied to solve linear and nonlinear systems of ordinary differential equations [15, 11, 16, 17, 18, 12, 19, 20, 7, 3, 21, 13, 22, 23] and it is applied to solve some biological equations in [24, 25].

Based on the assumption that the reader is familiar with DTM, most authors omit the proofs of some theorems. This assumption inspires the first part of this paper. Proofs of some important theorems which are often omitted are presented. These theorems are then extended with proofs. Moreso, we further show that the DTM converges to the exact solution when the problem involved is linear (whether homogeneous or nonhomogeneous).

Finally, we establish these proofs with some examples.

2 Differential Transform

The Differential Transform Method is an extension of the Taylor Expansion Method. The Taylor expansion for a function $y(x)$ about the point $x = x_0$ is defined as

$$y(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} \left. \frac{d^k}{dx^k} y(x) \right|_{x=a}. \tag{2.1}$$

Define the differential transform ([9, 10, 26]) of the function $y(x)$ about $x = a$ as

$$DT\left\{y\left(x\right)\right\} = Y_a\left[k\right] = \frac{1}{k!} \left. \frac{d^k}{dx^k} y(x) \right|_{x=a} \tag{2.2}$$

and the inverse transform as

$$DT^{-1}\left\{Y_a\left[k\right]\right\} = y\left(x\right) = \sum_{k=0}^{\infty} Y\left[k\right]\left(x - x_0\right)^k. \tag{2.3}$$

We also define the differential transform of $y(x)$ about $x = 0$ as

$$DT\left\{y\left(x\right)\right\} = Y\left[k\right] = \frac{1}{k!} \left. \frac{d^k}{dx^k} y(x) \right|_{x=0} \tag{2.4}$$

and the inverse transform as

$$DT^{-1}\left\{Y\left[k\right]\right\} = y\left(x\right) = \sum_{k=0}^{\infty} Y\left[k\right] x^k. \tag{2.5}$$

We define the Mth order approximation of $y(x)$ as

$$y\left(x\right)=\sum_{k=0}^{M}Y\left[k\right]x^{k}. \tag{2.6}$$

3 Theorems in Differential Transform

The following theorems and some of the proofs can be found in [9, 13, 10, 26, 27, 28, 29]. In the following theorems, we shall suppose α,β,γ,c are constants and that

$$DT\left\{y(x)\right\}=Y\left[k\right],\ DT\left\{a_j\left(x\right)\right\}=A_j\left[k\right],\ DT\left\{f(x)\right\}=F\left[k\right],\ DT\left\{g(x)\right\}=G\left[k\right],\ DT\left\{a\left(x\right)\right\}=A\left[k\right]$$

and define the delta function as

$$\delta_{k,n}=\begin{cases}0 & k\neq n\\ 1 & k=n\end{cases}. \tag{3.1}$$

Theorem 3.1. Linear combination. *Linear combination is closed under differential transform i.e. if $y(x)=\alpha f(x)\pm\beta g(x)$, then $Y[k]=\alpha F[k]+\beta G[k]$.*

Proof.

$$DT\left\{y(x)\right\} = \frac{1}{k!}\frac{d^k}{dx^k}\left(\alpha f\left(x\right)\pm\beta g\left(x\right)\right)\Big|_{x=0}=\alpha F\left[k\right]\pm\beta G\left[k\right].$$

□

Corollary 3.2. Scalar Multiplication. *If $y(x)=\alpha f(x)$, then $Y[k]=\alpha F[k]$.*

Proof. This follows from theorem 3.1 by setting $\beta=0$ □

Theorem 3.3. Polynomial function. *If $y\left(x\right)=cx^n$, then $Y\left[k\right]=c\delta_{n,k}$*

Proof. By definition

$$Y\left[k\right] = \frac{1}{k!}\frac{d^k}{dx^k}cx^n\Big|_{x=0}=c\begin{cases}1 & k=n\\ 0 & k\neq n\end{cases}=c\delta_{n,k}.$$

□

Corollary 3.4. Constant function. *If $y\left(x\right)=c$, then $Y\left[k\right]=c\delta_{0,k}$*

Proof. By setting $n=0$ in theorem 3.3, the proof is complete. □

Theorem 3.5. Exponential function. *If $y\left(x\right)=e^{ax+b}$, then $Y\left[k\right]=e^b\frac{a^k}{k!}$*

Proof. By definition

$$Y\left[k\right] = \frac{1}{k!}\frac{d^k}{dx^k}e^{ax+b}\Big|_{x=0}=e^b\frac{a^k}{k!}.$$

□

Theorem 3.6. Trigonometric functions. *If $y\left(x\right)=\sin\left(ax+b\right)$ and $y\left(x\right)=\cos\left(ax+b\right)$ then*

$$Y\left[k\right]=\frac{a^k}{k!}\sin\left(\frac{k\pi}{2}+b\right)\quad and\quad Y\left[k\right]=\frac{a^k}{k!}\cos\left(\frac{k\pi}{2}+b\right) \tag{3.2}$$

respectively.

Proof. By definition

$$Y[k] = \frac{1}{k!}\frac{d^k}{dx^k}\sin(ax+b)\bigg|_{x=0} = \frac{1}{k!}a^k\sin\left(ax+b+\frac{k\pi}{2}\right)\bigg|_{x=0} = \frac{a^k}{k!}\sin\left(\frac{k\pi}{2}+b\right).$$

and similarly,

$$Y[k] = \frac{a^k}{k!}\cos\left(\frac{k\pi}{2}+b\right).$$

□

Theorem 3.7. nth derivative. *If* $y(x) = \frac{d^n}{dx^n}f(x)$, *then*

$$Y[k] = \left(\prod_{r=1}^{n}(k+r)\right)F[k+n].$$

Proof. By definition

$$\begin{aligned} Y[k] &= \frac{1}{k!}\frac{d^k}{dx^k}\left(\frac{d^n}{dx^n}f(x)\right)\bigg|_{x=0} = \frac{1}{k!}\frac{d^{k+n}}{dx^{k+n}}f(x)\bigg|_{x=0} \\ &= \left(\prod_{r=1}^{n}(k+r)\right)\left(\frac{1}{(k+n)!}\frac{d^{k+n}}{dx^{k+n}}f(x)\bigg|_{x=0}\right) = \left(\prod_{r=1}^{n}(k+r)\right)F[k+n]. \end{aligned}$$

□

Corollary 3.8. *The differential transform of* $f'(x), f''(x), f''''(x)$ *are* $(k+1)F[k+1]$, $(k+1)(k+2)F[k+2]$, $(k+1)(k+2)(k+3)F[k+3]$ *respectively.*

Proof. These results are obtained by simply setting $n = 1, 2, 3$ respectively in theorem 3.7. □

Theorem 3.9. Convolution theorem. *If the convolution of* $F[k]$ *and* $G[k]$ *is defined as*

$$F[k]\otimes G[k] = \sum_{r=0}^{k}F[r]G[k-r]$$

then

$$DT\{f(x)g(x)\} = F[k]\otimes G[k] = G[k]\otimes F[k]$$

Proof. Let

$$y(x) = f(x)g(x)$$

then

$$\begin{aligned} Y[k] &= \frac{1}{k!}\frac{d^k}{dx^k}f(x)g(x)\bigg|_{x=0} = \sum_{r=0}^{k}\frac{1}{r!(k-r)!}\frac{d^r}{dx^r}f(x)\frac{d^{k-r}}{dx^{k-r}}g(x)\bigg|_{x=0} \\ &= \sum_{r=0}^{k}\frac{1}{r!}\frac{d^r}{dx^r}f(x)\bigg|_{x=0}\frac{1}{(k-r)!}\frac{d^{k-r}}{dx^{k-r}}g(x)\bigg|_{x=0} = \sum_{r=0}^{k}F[r]G[k-r] \end{aligned} \quad (3.3)$$

by interchanging the roles of f and g, we also have

$$Y[k] = \sum_{r=0}^{k}G[r]F[k-r] = \sum_{r=0}^{k}F[k-r]G[r]. \quad (3.4)$$

Hence,

$$DT\{f(x)g(x)\} = F[k]\otimes G[k] = G[k]\otimes F[k]. \quad (3.5)$$

□

Theorem 3.10. *Suppose*

$$y(x) = \prod_{i=1}^{n} f_i(x),$$

then

$$Y[k] = \sum_{r_1=0}^{k} \sum_{r_2=0}^{k_1} \cdots \sum_{r_{n-1}=0}^{k_{n-2}} F_1[r_1] F_2[r_2] \cdots F_{n-1}[r_{n-1}] F_n[k_{n-1}]$$

where

$$k_p = k - \prod_{i=1}^{p} r_i,\ 1 \leq p < n$$

Proof. By definition

$$Y[k] = \frac{1}{k!} \frac{d^k}{dx^k} \left(\prod_{i=1}^{n} f_i(x) \right) \Bigg|_{x=0}$$

so,

$$\begin{aligned} Y[k] &= \frac{1}{k!} \sum_{r_1=0}^{k} \left(\frac{k!}{k_1! r_1!} \frac{d^{r_1}}{dx^{r_1}} f_1(x) \Big|_{x=0} \frac{d^{k_1}}{dx^{k_1}} \left(\prod_{i=2}^{n} f_i(x) \right) \Bigg|_{x=0} \right) \\ &= \sum_{r_1=0}^{k} F_1[r_1] \left(\frac{1}{k_1!} \frac{d^{k_1}}{dx^{k_1}} \left(\prod_{i=2}^{n} f_i(x) \right) \Bigg|_{x=0} \right) \end{aligned}$$

repeating this process again, we have

$$Y[k] = \sum_{r_1=0}^{k} F_1[r_1] \sum_{r_2=0}^{k_2} F_2[r_2] \left(\frac{1}{k_2!} \frac{d^{k_2}}{dx^{k_2}} \left(\prod_{i=3}^{n} f_i(x) \right) \Bigg|_{x=0} \right)$$

and continuing in this manner for $n-1$ times, we have

$$\begin{aligned} Y[k] &= \sum_{r_1=0}^{k} F_1[r_1] \sum_{r_2=0}^{k_2} F_2[r_2] \cdots \sum_{r_{n-1}=0}^{k_{n-1}} F_{n-1}[r_{n-1}] \left(\frac{1}{k_{n-1}!} \frac{d^{k_{n-1}}}{dx^{k_{n-1}}} (f_n(x)) \Big|_{x=0} \right) \\ &= \sum_{r_1=0}^{k} F_1[r_1] \sum_{r_2=0}^{k_2} F_2[r_2] \cdots \sum_{r_{n-1}=0}^{k_{n-1}} F_{n-1}[r_{n-1}] F[k_{n-1}] \\ &= \sum_{r_1=0}^{k} \sum_{r_2=0}^{k_2} \cdots \sum_{r_{n-1}=0}^{k_{n-1}} F_1[r_1] F_2[r_2] \cdots F_{n-1}[r_{n-1}] F[k_{n-1}]. \end{aligned}$$

□

Theorem 3.11. *Suppose*

$$y(x) = \frac{d^n}{dx^n} f(x) \frac{d^m}{dx^m} g(x)$$

then

$$Y[k] = \sum_{r=0}^{k} \left(\prod_{i=1}^{n} (r+i) \right) \left(\prod_{j=1}^{m} (k-r+j) \right) F[r+n] G[k-r+m].$$

Proof. By definition

$$\begin{aligned} Y[k] &= \frac{1}{k!}\frac{d^k}{dx^k}\left(\frac{d^n}{dx^n}f(x)\,\frac{d^m}{dx^m}g(x)\right) \\ &= \frac{1}{k!}\sum_{r=0}^{k} {}^kC_r\frac{d^r}{dx^r}\left(\frac{d^n}{dx^n}f(x)\right)\frac{d^{k-r}}{dx^{k-r}}\left(\frac{d^m}{dx^m}g(x)\right) \\ &= \sum_{r=0}^{k}\left(\frac{1}{r!}\frac{d^{n+r}}{dx^{n+r}}f(x)\right)\left(\frac{1}{(k-r)!}\frac{d^{m+k-r}}{dx^{m+k-r}}g(x)\right) \end{aligned}$$

thus,

$$\begin{aligned} Y[k] &= \sum_{r=0}^{k}\left(\prod_{i=1}^{n}(r+i)\right)\left(\prod_{j=1}^{m}(k-r+j)\right)\left(\frac{1}{(r+n)!}\frac{d^{n+r}}{dx^{n+r}}f(x)\right)\left(\frac{1}{(k-r+m)!}\frac{d^{k-r+m}}{dx^{k-r+m}}g(x)\right) \\ &= \sum_{r=0}^{k}\left(\prod_{i=1}^{n}(r+i)\right)\left(\prod_{j=1}^{m}(k-r+j)\right)F[r+n]\,G[k-r+m] \end{aligned}$$

□

Theorem 3.12. *If* $y(x) = (1+x)^m$ *then* $Y[k] =^m C_k$

Proof. By definition

$$Y[k] = \frac{1}{k!}\frac{d^k}{dx^k}(1+x)^m\Big|_{x=0} = {}^mC_k\,(1+x)^{m-k}\Big|_{x=0} = {}^mC_k.$$

□

Theorem 3.13. *If*

$$y(x) = \int_0^x f(t)dt$$

then

$$Y[k] = \begin{cases} 0 & k = 0 \\ \frac{F[k-1]}{k} & k > 0 \end{cases}.$$

Proof. Let $\bar{f}(x)$ be the anti-derivative of $f(x)$, then

$$\int_0^x f(t)dt = \bar{f}(x) - \bar{f}(0)$$

and

$$Y[k] = \frac{1}{k!}\frac{d^k}{dx^k}\left(\int_0^x f(t)dt\right)\Big|_{x=0} = \frac{1}{k!}\frac{d^k}{dx^k}\left(\bar{f}(x) - \bar{f}(0)\right)\Big|_{x=0}$$

When $k = 0$,

$$Y[0] = \bar{f}(0) - \bar{f}(0) = 0.$$

and when $k > 0$, we have

$$Y[k] = \frac{1}{k!}\frac{d^k}{dx^k}\left(\bar{f}(x) - \bar{f}(0)\right)\Big|_{x=0} = \frac{1}{k!}\frac{d^{k-1}}{dx^{k-1}}f(x)\Big|_{x=0} = \frac{F[k-1]}{k}.$$

□

Theorem 3.14. *If* $y(x) = f(ax)$ *then*

$$Y[k] = a^k F[k].$$

Proof. By using the inverse differential transform, we have

$$f(x) = \sum_{k=0}^{\infty} F[k]\, x^k$$

and clearly,

$$f(ax) \quad = \quad \sum_{k=0}^{\infty} F[k]\,(ax)^k \;=\; \sum_{k=0}^{\infty} a^k F[k]\, x^k$$

and the differential transform is

$$Y[k] = DT\{f(ax)\} = a^k F[k].$$

□

Theorem 3.15. $Y[k]$ *and* $Y_a[k]$ *are related by the relations*

$$Y[k] = \sum_{r=k}^{\infty} {}^{r}C_k \cdot Y_a[r]\,(-a)^{r-k} \quad \textit{and } Y_a[k] = \sum_{r=k}^{\infty} {}^{r}C_k \cdot Y[r]\, a^{r-k}$$

Proof. If

$$Y_a[k] = \frac{1}{k!}\,\frac{d^k}{dx^k}\, y(x)\bigg|_{x=a}, \tag{3.6}$$

then the inverse differential transform is

$$y(x) = \sum_{k=0}^{\infty} Y_a[k]\,(x-a)^k.$$

By substituting the Binomial expansion, we have

$$\begin{aligned} y(x) \quad &= \quad \sum_{k=0}^{\infty} Y_a[k] \sum_{r=0}^{k} {}^{k}C_r x^r\,(-a)^{k-r} \;=\; \sum_{k=0}^{\infty}\sum_{r=0}^{k} {}^{k}C_r \cdot Y_a[k]\, x^r\,(-a)^{k-r} \\ &= \quad \sum_{k=0}^{\infty}\sum_{r=k}^{\infty} {}^{r}C_k \cdot Y_a[r]\, x^k\,(-a)^{r-k} \;=\; \sum_{k=0}^{\infty}\left(\sum_{r=k}^{\infty} {}^{r}C_k \cdot Y_a[r]\,(-a)^{r-k}\right) x^k \end{aligned}$$

and therefore

$$Y[k] = \sum_{r=k}^{\infty} {}^{r}C_k \cdot Y_a[r]\,(-a)^{r-k}.$$

Similarly,

$$y(x) \quad = \quad \sum_{k=0}^{\infty} Y[k]\, x^k \;=\; \sum_{k=0}^{\infty} Y[k]\,(x-a+a)^k$$

By substituting the Binomial expansion, we have

$$\begin{aligned} y(x) \quad &= \quad \sum_{k=0}^{\infty} Y[k] \sum_{r=0}^{k} {}^{k}C_r\,(x-a)^r\, a^{k-r} \;=\; \sum_{k=0}^{\infty}\sum_{r=0}^{k} {}^{k}C_r \cdot Y[k]\,(x-a)^r\, a^{k-r} \\ &= \quad \sum_{k=0}^{\infty}\sum_{r=k}^{\infty} {}^{r}C_k \cdot Y[r]\,(x-a)^k\, a^{r-k} \;=\; \sum_{k=0}^{\infty}\left(\sum_{r=k}^{\infty} {}^{r}C_k \cdot Y[r]\, a^{r-k}\right)(x-a)^k \end{aligned}$$

and therefore

$$Y_a\,[k] = \sum_{r=k}^{\infty} {}^rC_k \cdot Y\,[r]\,(-a)^{r-k}\,.$$

□

4 Main Results

Theorem 4.1. *If*

$$y(x) = \int_0^x f\,(t)\,g\,(t)\,dt$$

then

$$Y\,[k] = \begin{cases} 0 & k = 0 \\ \frac{1}{k}\sum_{r=0}^{k-1} F\,[r]\,G[k-1-r] & k > 0 \end{cases}.$$

Proof. Let $\bar{y}\,(x)$ be the anti-derivative of $f\,(x)\,g(x)$, then

$$\int_0^x f\,(t)\,g\,(t)\,dt = \bar{y}\,(x) - \bar{y}\,(0)$$

and

$$\begin{aligned} Y\,[k] &= \frac{1}{k!}\,\frac{d^k}{dx^k}\int_0^x f\,(t)\,g\,(t)\,dt\Big|_{x=0} \\ &= \frac{1}{k!}\,\frac{d^k}{dx^k}\,(\bar{y}\,(x) - \bar{y}\,(0))\Big|_{x=0} \end{aligned}$$

When $k = 0$,

$$Y\,[0] = \bar{y}\,(0) - \bar{y}\,(0) = 0.$$

and when $k > 0$, we have

$$\begin{aligned} Y\,[k] &= \frac{1}{k!}\,\frac{d^k}{dx^k}\,(\bar{y}\,(x) - \bar{y}\,(0))\Big|_{x=0} \\ &= \frac{1}{k}\,\frac{1}{(k-1)!}\,\frac{d^{k-1}}{dx^{k-1}} f\,(x)\,g\,(x)\Big|_{x=0} \end{aligned}$$

By applying the convolution theorem 3.9, we have

$$Y\,[k] = \frac{1}{k}\sum_{r=0}^{k-1} F\,[r]\,G\,[k-1-r]$$

□

Theorem 4.2. *Initial conditions. If*

$$\frac{d^n}{dx^n}y(x)\Big|_{x=0} = \alpha \;\; \textit{then}\; Y\,[n] = \frac{\alpha}{n!}.$$

Proof. By definition

$$Y\,[k] = \frac{1}{k!}\,\frac{d^k}{dx^k}y(x)\Big|_{x=0}$$

and so, when $k = n$, we have

$$Y[n] = \frac{1}{n!}\left.\frac{d^n}{dx^n}y(x)\right|_{x=0}$$

and inserting the initial condition, we have

$$Y[n] = \frac{\alpha}{n!}.$$

□

Corollary 4.3. *If* $y(0) = \alpha_0,\ y'(0) = \alpha_1,\ y''(0) = \alpha_2,\ y'''(0) = \alpha_3,$ *then* $Y[0] = \alpha_0,\ Y[1] = \alpha_1,$ $Y[2] = \frac{\alpha_2}{2!}$ *and* $Y[3] = \frac{\alpha_3}{3!}.$

Proof. This is a direct consequence of theorem 4.2 by simply setting $n = 0, 1, 2, 3$ respectively. □

Theorem 4.4. *Boundary conditions. If*

$$y(b) = \beta \ \ \textit{then} \ \ \beta = \sum_{k=0}^{\infty} Y[k]\, b^k.$$

Proof. By definition of the inverse differential transform

$$y(x) = \sum_{k=0}^{\infty} Y[k]\, x^k$$

and so, by substituting $x = b$, we have

$$\beta = \sum_{k=0}^{\infty} Y[k]\, b^k.$$

□

Remark 4.1. In application, we take the Mth order approximation and we therefore set

$$\beta = \sum_{k=0}^{M} Y[k]\, b^k.$$

Theorem 4.5. *If* a, b *are constants in the first order ordinary differential equation,*

$$a\frac{dy}{dx} + by = 0,\ y(0) = \alpha, \tag{4.1}$$

then the differential transform method converges to the exact solution.

Proof. The differential equation has a general solution

$$y(x) = \alpha e^{-bx/a}$$

Taking the differential transform of equation 4.1, we have

$$a(k+1)Y[k+1] + bY[k] = 0,\ Y[0] = \alpha$$

and on rearranging, we have

$$Y[k+1] = -\frac{b}{a(k+1)}Y[k].$$

Solving this recurrence relation, we have

$$Y[k] = \frac{1}{k!}\left(\frac{-b}{a}\right)^k \alpha$$

and thus, the solution obtained from the differential transform method is

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-b}{a}\right)^k \alpha x^k \\ &= \alpha \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-bx}{a}\right)^k \\ &= \alpha e^{-bx/a}. \end{aligned}$$

□

Theorem 4.6. *If a, b, c are constants in the second order ordinary differential equation,*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,\ y(0) = \alpha_0,\ y'(0) = \alpha_1 \tag{4.2}$$

then the differential transform method converges to the exact solution.

Proof. Taking the differential transform of equation 4.2, we have

$$a(k+1)(k+2)Y[k+2] + b(k+1)Y[k+1] + cY[k] = 0,\ Y[0] = \alpha_0, Y[1] = \alpha_1 \tag{4.3}$$

and the differential transform solution is

$$y(x) = \sum_{k=0}^{\infty} Y[k]\, x^k. \tag{4.4}$$

Substituting the series 4.4 in the original problem 4.2, we have

$$a\sum_{k=2}^{\infty} k(k-1)Y[k]\,x^{k-2} + b\sum_{k=2}^{\infty} kY[k]\,x^{k-1} + c\sum_{k=2}^{\infty} Y[k]\,x^k = 0$$

and on rearranging, we have

$$\sum_{k=0}^{\infty} \left(a(k+1)(k+2)Y[k+2] + b(k+1)Y[k+1] + cY[k]\right)x^k = 0,$$

and comparing coefficients gives

$$a(k+1)(k+2)Y[k+2] + b(k+1)Y[k+1] + cY[k] = 0. \tag{4.5}$$

Hence, the recurrence relation from the DTM is satisfied. □

Theorem 4.7. *If a_i's ($\forall\ i = 1, 2, \cdots, n$) are constants in the nth order homogeneous ordinary differential equation,*

$$a_0 y + \sum_{r=1}^{n} a_r \frac{d^r y}{dx^r} = 0,\ y(0) = \alpha_0,\ \left.\frac{d^i}{dx^i} y(x)\right|_{x=0} = \alpha_i,\ i = 1, 2, \cdots, n-1 \tag{4.6}$$

then the differential transform method converges to the exact solution.

Proof. Taking the differential transform of equation 4.6, we have

$$a_0Y[k] + \sum_{r=1}^{n} a_r \left(\prod_{i=1}^{r} (k+i)\right) Y\left[k+r\right] = 0, \quad Y\left[0\right] = \alpha_0,\ Y\left[i\right] = \frac{\alpha_i}{i!}, i = 1, 2, \cdots, n-1 \tag{4.7}$$

and the differential transform solution is

$$y\left(x\right) = \sum_{k=0}^{\infty} Y\left[k\right] x^k. \tag{4.8}$$

Substituting the series 4.8 in the original problem 4.6, we have

$$\begin{aligned} 0 &= a_0 \sum_{k=0}^{\infty} Y\left[k\right] x^k + \sum_{r=1}^{n} a_r \frac{d^r}{dx^r} \left(\sum_{k=0}^{\infty} Y\left[k\right] x^k\right) \\ &= a_0 \sum_{k=0}^{\infty} Y\left[k\right] x^k + \sum_{r=1}^{n} a_r \sum_{k=r}^{\infty} Y\left[k\right] \left(\prod_{i=0}^{r-1} (k-i)\right) x^{k-r} \\ &= a_0 \sum_{k=0}^{\infty} Y\left[k\right] x^k + \sum_{k=r}^{\infty} \sum_{r=1}^{n} a_r Y\left[k\right] \left(\prod_{i=0}^{r-1} (k-i)\right) x^{k-r} \\ &= a_0 \sum_{k=0}^{\infty} Y\left[k\right] x^k + \sum_{k=0}^{\infty} \sum_{r=1}^{n} a_r Y\left[k+r\right] \left(\prod_{i=0}^{r-1} (k+r-i)\right) x^k \\ &= \sum_{k=0}^{\infty} \left(a_0 Y\left[k\right] + \sum_{r=1}^{n} a_r \left(\prod_{i=1}^{r} (k+i)\right) Y\left[k+r\right]\right) x^k \end{aligned}$$

and comparing coefficients gives

$$a_0 Y\left[k\right] + \sum_{r=1}^{n} a_r \left(\prod_{i=1}^{r} (k+i)\right) Y\left[k+r\right] = 0. \tag{4.9}$$

Hence, the recurrence relation from the DTM is satisfied. □

Theorem 4.8. *For a variable coefficient nth order homogeneous ordinary differential equation*

$$a_0\left(x\right) y + \sum_{r=1}^{n} a_r\left(x\right) \frac{d^r y}{dx^r} = 0,\ y(0) = \alpha_0,\ \frac{d^i}{dx^i} y\left(x\right)\Big|_{x=0} = \alpha_i,\ i = 1, 2, \cdots, n-1 \tag{4.10}$$

the differential transform method converges to the exact solution.

Proof. Taking the differential transform of equation 4.10, we have

$$\sum_{m=0}^{k} A_0\left[m\right] Y[k-m] + \sum_{r=1}^{n} \left(\sum_{m=0}^{k} A_r[m] \left(\prod_{j=1}^{r} (k-m+j)\right) Y\left[k-m+r\right]\right) = 0, \tag{4.11}$$

subject to

$$Y\left[0\right] = \alpha_0, Y\left[i\right] = \frac{\alpha_i}{i!}, i = 1, 2, \cdots, n-1.$$

Rearranging, we have

$$\sum_{m=0}^{k} \left(A_0\left[m\right] Y[k-m] + \sum_{r=1}^{n} A_r[m] Y\left[k-m+r\right] \left(\prod_{j=1}^{r} (k-m+j)\right)\right) = 0.$$

Substituting the differential transforms

$$y(x) = \sum_{k=0}^{\infty} Y[k]\, x^k, \quad a_r(x) = \sum_{k=0}^{\infty} A_r[k]\, x^k. \tag{4.12}$$

into the original problem 4.10, we have

$$\begin{aligned}
0 &= \sum_{k=0}^{\infty} A_0[k]\, x^k \sum_{k=0}^{\infty} Y[k]\, x^k + \sum_{r=1}^{n} \left(\sum_{k=0}^{\infty} A_r[k]\, x^k \cdot \frac{d^r}{dx^r} \left(\sum_{k=0}^{\infty} Y[k]\, x^k \right) \right) \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} A_0[m]\, Y[k-m]\, x^k + \sum_{r=1}^{n} \left(\sum_{k=0}^{\infty} A_r[k]\, x^k \cdot \sum_{m=0}^{\infty} Y[m+r] \left(\prod_{j=0}^{r-1} (m+r-j) \right) x^m \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} A_0[m]\, Y[k-m]\, x^k + \sum_{r=1}^{n} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left(A_r[m]\, Y[k-m+r] \left(\prod_{j=0}^{r-1} (k-m+r-j) \right) \right) x^k \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left(A_0[m]\, Y[k-m] + \sum_{r=1}^{n} A_r[m]\, Y[k-m+r] \left(\prod_{j=0}^{r-1} (k-m+r-j) \right) \right) x^k \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left(A_0[m]\, Y[k-m] + \sum_{r=1}^{n} A_r[m]\, Y[k-m+r] \left(\prod_{j=1}^{r} (k-m+j) \right) \right) x^k
\end{aligned}$$

and comparing coefficients, we have

$$\sum_{m=0}^{k} \left(A_0[m]\, Y[k-m] + \sum_{r=1}^{n} A_r[m]\, Y[k-m+r] \left(\prod_{j=1}^{r} (k-m+j) \right) \right) = 0. \tag{4.13}$$

Hence, the recurrence relation from the DTM is satisfied. □

Theorem 4.9. *If a_i's ($\forall\, i = 1, 2, \cdots, n$) are constants in the nth order nonhomogeneous ordinary differential equation,*

$$a_0 y + \sum_{r=1}^{n} a_r \frac{d^r y}{dx^r} = f(x)\,, \ y(0) = \alpha_0, \ \frac{d^i}{dx^i} y(x) \bigg|_{x=0} = \alpha_i, \ i = 1, 2, \cdots, n-1 \tag{4.14}$$

then the differential transform method converges to the exact solution.

Proof. Taking the differential transform of equation 4.14, we have

$$a_0 Y[k] + \sum_{r=1}^{n} a_r \left(\prod_{i=1}^{r} (k+i) \right) Y[k+r] = F[k]\,, \ Y[0] = \alpha_0, Y[i] = \frac{\alpha_i}{i!}, i = 1, 2, \cdots, n-1 \tag{4.15}$$

Substituting the differential transforms

$$y(x) = \sum_{k=0}^{\infty} Y[k]\, x^k, \ f(x) = \sum_{k=0}^{\infty} F[k]\, x^k. \tag{4.16}$$

in the original problem 4.14, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} F[k]\,x^k &= a_0\sum_{k=0}^{\infty} Y[k]\,x^k + \sum_{r=1}^{n} a_r \frac{d^r}{dx^r}\left(\sum_{k=0}^{\infty} Y[k]\,x^k\right)\\
&= a_0\sum_{k=0}^{\infty} Y[k]\,x^k + \sum_{r=1}^{n} a_r \sum_{k=r}^{\infty} Y[k]\left(\prod_{i=0}^{r-1}(k-i)\right)x^{k-r}\\
&= a_0\sum_{k=0}^{\infty} Y[k]\,x^k + \sum_{k=r}^{\infty}\sum_{r=1}^{n} a_r Y[k]\left(\prod_{i=0}^{r-1}(k-i)\right)x^{k-r}\\
&= a_0\sum_{k=0}^{\infty} Y[k]\,x^k + \sum_{k=0}^{\infty}\sum_{r=1}^{n} a_r Y[k+r]\left(\prod_{i=0}^{r-1}(k+r-i)\right)x^{k}\\
&= a_0\sum_{k=0}^{\infty} Y[k]\,x^k + \sum_{k=0}^{\infty}\left(\sum_{r=1}^{n} a_r Y[k+r]\prod_{i=0}^{r-1}(k+r-i)\right)x^{k}\\
&= \sum_{k=0}^{\infty}\left(a_0 Y[k] + \sum_{r=1}^{n} a_r Y[k+r]\prod_{i=0}^{r-1}(k+r-i)\right)x^{k}\\
&= \sum_{k=0}^{\infty}\left(a_0 Y[k] + \sum_{r=1}^{n} a_r \left(\prod_{i=1}^{r}(k+i)\right) Y[k+r]\right)x^{k}
\end{aligned}$$

and comparing coefficients gives

$$a_0 Y[k] + \sum_{r=1}^{n} a_r \left(\prod_{i=1}^{r}(k+i)\right) Y[k+r] = F[k]\,. \tag{4.17}$$

Hence, the recurrence relation from the DTM is satisfied. □

Theorem 4.10. *For a variable coefficient nth order nonhomogeneous ordinary differential equation*

$$a_0(x)\,y + \sum_{r=1}^{n} a_r(x)\frac{d^r y}{dx^r} = f(x)\,,\ y(0)=\alpha_0,\ \frac{d^i}{dx^i}y(x)\Big|_{x=0} = \alpha_i,\ i=1,2,\cdots,n-1 \tag{4.18}$$

the differential transform method converges to the exact solution.

Proof. Taking the differential transform of equation 4.18, we have

$$\sum_{m=0}^{k} A_0[m]\,Y[k-m] + \sum_{r=1}^{n}\left(\sum_{m=0}^{k} A_r[m]\left(\prod_{j=1}^{r}(k-m+j)\right)Y[k-m+r]\right) = F[k]\,, \tag{4.19}$$

subject to

$$Y[0]=\alpha_0, Y[i] = \frac{\alpha_i}{i!}, i=1,2,\cdots,n-1.$$

Rearranging, we have

$$\sum_{m=0}^{k}\left(A_0[m]\,Y[k-m] + \sum_{r=1}^{n} A_r[m] Y[k-m+r]\left(\prod_{j=1}^{r}(k-m+j)\right)\right) = F[k]\,.$$

Substituting the differential transform 4.20

$$y(x) = \sum_{k=0}^{\infty} Y[k]\,x^k,\ a_r(x) = \sum_{k=0}^{\infty} A_r[k]\,x^k,\ f(x) = \sum_{k=0}^{\infty} F[k]\,x^k. \tag{4.20}$$

in the original problem 4.18, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} F[k]\, x^k &= \sum_{k=0}^{\infty} A_0[k]\, x^k \sum_{k=0}^{\infty} Y[k]\, x^k + \sum_{r=1}^{n} \left(\sum_{k=0}^{\infty} A_r[k]\, x^k \cdot \frac{d^r}{dx^r} \left(\sum_{k=0}^{\infty} Y[k]\, x^k \right) \right) \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} A_0[m]\, Y[k-m]\, x^k + \sum_{r=1}^{n} \left(\sum_{k=0}^{\infty} A_r[k]\, x^k \cdot \sum_{m=0}^{\infty} Y[m+r] \left(\prod_{j=0}^{r-1} (m+r-j) \right) x^m \right) \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} A_0[m]\, Y[k-m]\, x^k + \sum_{r=1}^{n} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left(A_r[m]\, Y[k-m+r] \left(\prod_{j=0}^{r-1} (k-m+r-j) \right) \right) x^k \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left(A_0[m]\, Y[k-m] + \sum_{r=1}^{n} A_r[m]\, Y[k-m+r] \left(\prod_{j=0}^{r-1} (k-m+r-j) \right) \right) x^k \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left(A_0[m]\, Y[k-m] + \sum_{r=1}^{n} A_r[m]\, Y[k-m+r] \left(\prod_{j=1}^{r} (k-m+j) \right) \right) x^k
\end{aligned}$$

and comparing coefficients, we have

$$\sum_{m=0}^{k} \left(A_0[m]\, Y[k-m] + \sum_{r=1}^{n} A_r[m]\, Y[k-m+r] \left(\prod_{j=1}^{r} (k-m+j) \right) \right) = F[k]. \qquad (4.21)$$

Hence, the recurrence relation from the DTM is satisfied. □

5 Numerical Examples

Example 5.1. *Consider the homogeneous second order linear ordinary differential equation*

$$y'' + 5y' + 6y = 0;\ y(0) = 0, y'(0) = 1.$$

The general solution is

$$y = e^{-2x} - e^{-3x} = \sum_{k=0}^{\infty} \frac{\left((-2)^k - (-3)^k\right) x^k}{k!}$$

and to $O(7)$,we have

$$y \quad = \quad x - \frac{5}{2}x^2 + \frac{19}{6}x^3 - \frac{65}{24}x^4 + \frac{211}{120}x^5 - \frac{133}{144}x^6 + O(x^7) \qquad (5.1)$$

We take the Differential Transform and we get

$$(k+1)(k+2)\, Y[k+2] + 5(k+1)\, Y[k+1] + 6Y[k] = 0,$$

subject to

$$Y[0] = 0,\ Y[1] = 1.$$

By rearranging, we have

$$Y[k+2] = -\frac{5Y[k+1]}{(k+2)} - \frac{6Y[k]}{(k+1)(k+2)},$$

so that the 6th order approximation is

$$y(x) \quad = \quad \sum_{k=0}^{M} Y[k]\, x^k \ = \ x - \frac{5}{2}x^2 + \frac{19}{6}x^3 - \frac{65}{24}x^4 + \frac{211}{120}x^5 - \frac{133}{144}x^6 + \cdots$$

Clearly, this series converges to the exact solution as shown in equation 5.1.

Example 5.2. *Consider the equation the nonhomogeneous second order linear ordinary differential equation*

$$y'' + 5y' + 6y = e^x;\ y(0) = 0, y'(0) = 1.$$

The general solution is

$$\begin{aligned} y &= \frac{1}{12}\left(8e^{-2x} - 9e^{-3x} + e^x\right) \\ &= x - 2x^2 + \frac{5}{2}x^3 - \frac{25}{12}x^4 + \frac{161}{120}x^5 - \frac{7}{10}x^6 + \cdots \end{aligned}$$

We now proceed to take the differential transform of the problem

$$(k+1)(k+2)Y[k+2] + 5(k+1)Y[k+1] + 6Y[k] = \frac{1}{k!},\ Y[0] = 0, Y[1] = 1$$

on rearranging, we have

$$Y[k+2] = \frac{1}{(k+2)!} - \frac{5Y[k+1]}{(k+2)} - \frac{6Y[k]}{(k+1)(k+2)}.$$

So that

$$Y[2] = -2,\ Y[3] = \frac{5}{2},\ Y[4] = -\frac{25}{12},\ Y[5] = \frac{161}{120},\ Y[6] = -\frac{7}{10}, \cdots$$

and thus we have the solution as

$$y = x - 2x^2 + \frac{5}{2}x^3 - \frac{25}{12}x^4 + \frac{161}{120}x^5 - \frac{7}{10}x^6 + \cdots$$

which converges to the exact solution.

Example 5.3. *Consider the nonhomogeneous first order linear ordinary differential equation*

$$(1+x)\frac{dy}{dx} + y = (1+x)e^x,\ y(0) = 0.$$

The exact solution is

$$\begin{aligned} y &= x(1+x)^{-1}e^x = x\sum_{n=0}^{\infty}(-x)^n\sum_{m=0}^{\infty}\left(\frac{x^m}{m!}\right) \\ &= x\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{(-1)^m}{(n-m)!}\right)x^n = x + \frac{1}{2}x^3 - \frac{1}{3}x^4 + \frac{3}{8}x^5 + \cdots \end{aligned}$$

6 Conclusion

The first part of this work is dedicated to proving some theorems whose proofs have been long ignored. Most authors assume the knowledge of these theorems, so they do not bother to prove the theorems. The theorems are therefore proved to serve as a reference for any work that would want to use the theorems without proofs.

The later part of this work establishes the convergence of the solution obtained from the DTM to the exact solution. The DTM solution converges to the exact solution for any ordinary differential equation, whether homogeneous or nonhomogeneous. These theorems are illustrated with some examples.

The differential transform method reduces the difficulty of solving an ordinary differential equation to a simple recursive equation that are relatively easy to solve. This work establishes that without solving a differential equation to get a closed form solution, we can obtain an approximation (up to any term of interest) to the solution by solving using the DTM.

It is important to mention here that since DTM is a method derived from the Taylors' expansion method and Taylors' expansion is only valid for functions that are continuous in the region of concern, we note that this method will as well be useful only for solutions that are continuous in the region under consideration.

In addition to this, we recognise that the solution is assumed to admit a Taylors' expansion.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Biazar J, Babolian E, Islam R. Solution of a system of volterra integral equations of the first kind by adomian method. Appl. Math. Comput. 2003;139:249-258.

[2] El-Sayed S. The decomposition method for studying the klein-gordon equation. Chaos, Solitons & Fractals. 2003;18:1025-1030.

[3] Hassan OA-H. Comparison differential transform technique with adomian decomposition method for linear and nonlinear initial value problems. Chaos, Solitons & Fractals. 2008;36:53-65.

[4] Jafari H, Gejji VD. Revised adomian decomposition method for solving system of ordinary and fractional differential equations. Appl. Math. Comput. 2006;181:598-608.

[5] Biazar J, Babolian E, Islam R. Solution of the system of ordinary differential equations by adomian decomposition method. Appl. Math. Comput. 2004;147(3):713-719.

[6] Shawagfeh NT, Kaya D. Comparing numerical methods for solutions of ordinary differential equations. Appl. Math. Lett. 2004;17:323-328.

[7] Biazar J, Hes HG. Variational iteration method for solving linear and nonlinear and system of ordinary differential equations. Appl. Math. Comput. 2007;191:287-297.

[8] Yusufoglu OE. The variational iteration method for studying the klein-gordon equation. Appl. Math. Lett. 2008;(21):669-674.

[9] Zhou J. Differential transformation and application for electrical circuits. Huazhong University Press, Wuhan, China; 1986.

[10] Chen CL, Lin SH, Chen C. Application of taylor transformation to nonlinear predictive control problem. Appl. Math. Model. 1996;20:699-710.

[11] Chen CK, Ho SH. Solving partial differential equations by two-dimensional differential transform method. Appl. Math. Comput. 1999;106:171-179.

[12] Ayaz F. Solution of the system of differential equations by differential transform method. Appl. Math. Comput. 2004;147:547-567.

[13] Odibat Z. Differential transform method for solving volterra integral equation with separable kernels. Math. Comput. Model. 2008;48:1144-1146.

[14] Javidi M, Golbabai A. Modified homotopy perturbation method for solving non-linear fredholm integral equations. Chaos, Solutions and Fractals; 2007.

[15] Kangalgil OF, Ayaz OF. Solitary wave solutions for the kdv and mkdv equations by differential transform method. Chaos, Solitons & Fractals; 2008.

[16] Jang MJ, Chen CL, Liu Y. Two-dimensional differential transform for partial differential equations. Appl. Math. Comput. 2001;121:261-270.

[17] Hassan OA-H. Different applications for the differential transformation in the differential equations. Appl. Math. Comput. 2002;129:183-201.

[18] Ayaz F. On two-dimensional differential transform method. Appl. Math. Comput. 2003;143:361-374.

[19] Arikoglu A, Ozkol I. Solution of boundary value problems for integro-differential equations by using transform method. Appl. Math. Comput. 2005;168:1145-1158.

[20] Kurnaz A, Oturnaz G, Kiris M. n-dimensional differential transformation method for solving linear and nonlinear pdes. Int. J. Comput. Math. 2005;82:369-380.

[21] Kanth ASVR, Aruna K. Differential transform method for solving linear and non-linear systems of partial differential equations. Phys. Lett. 2008;A 372:6896-6898.

[22] Kanth ASVR, Aruna K. Differential transform method for solving the linear and nonlinear klein-gordon equation. Computer Physics Communications. 2009;180:708-711.

[23] Mirzaee F. Differential transform method for solving linear and nonlinear systems of ordinary differential equations. Applied Mathematical Sciences. 2011;5(70):3465-3472.

[24] Tari A. The differential transform method for solving the model describing biological species living together. Iranian Journal of Mathematical Sciences and Informatics. 2012;7:(2):63-74.

[25] Moghadam MM, Saeedi H. Application of differential transforms for solving the volterra integro-partial differential equations. Iranian Journal of Science & Technology, Transaction. 2010;A 34:(A1).

[26] Jang MJ, Chen CL, Liu YC. On solving the initial value problems using the differential transformation method. Appl. Math. Comput. 2000;115:145-160.

[27] Abbasov A, Bahadir AR. The investigation of the transient regimes in the nonlinear systems by the generalized classical method. Math. Prob. Eng. 2005;5:503-519.

[28] Hassan IHA-H. Differential transformation technique for solving higher-order initial value problems. Appl. Math. Comput. 2004;154:299-311.

[29] Malik M, Dang HH. Vibration analysis of continuous system by differential transformation. Appl. Math. Comput. 1998;96:17-26.

© 2017 Oke; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/21322