British Journal of Mathematics & Computer Science
2(3): 176-186, 2012

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Some New Generalizations of Steffensen's Inequality

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Research Article

Received: 31st March 2012
Accepted: 14th October 2012
Published: 1st December 2012

Abstract

Extensions and new inequalities concerning Steffensen's inequality are presented.

2000 Mathematics Subject Classification: 26D15, 26D10

Keywords: Steffensen's inequality, integral inequality.

1 Introduction

Steffensen's inequality reads as follows:

Theorem 1.1 *Assume that two integrable functions* f *and* g *are defined on the interval* (a,b), *that is* f *non- increasing and that* $0 \le g(t) \le 1$ *in* (a,b). *Then*

$$\int_{b-\lambda}^{b} f(t)\,dt \le \int_{a}^{b} f(t)\,g(t)\,dt \le \int_{a}^{a+\lambda} f(t)\,dt, \qquad (1)$$

where $\lambda = \int_{a}^{b} g(t)\,dt$.

As an example to Steffensen's inequality, we can take the following:

$$f(t) = \frac{1}{1+t^2}, \quad g(t) = t, \quad a = 0, \quad b = 1, \text{ to have}$$

$$0.32 < 0.34 < 0.46.$$

Bellman(1950) gives the following generalization via new proof .

Theorem 1.2 Let $f(t)$ *be a non-negative and monotone decreasing in* $[a,b]$ *and*

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$\in L^p[a,b\}$ *and let* $g(t) \geq 0$ *in* $[a,b]$ *and* $\int_a^b g(t)\,dt \leq 1,$ *where* $p > 1$ *and* $(1/p) + (1/q) = 1$. *Then*

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \leq \int_a^{a+\mathrm{a}} f^{\,p}(t)\,dt \qquad \left(\mathrm{a} = \left(\int_a^b g(t)\,dt\right)\right)^p. \tag{2}$$

It may be mentioned that Bellman’s result is not correct as has been mentioned by Godunova and Levin (1968), A generalization in a different sense is made for $p \leq 1$. Inequality for $p \geq 1$, which similar to inequality (2) is given in Berch ((1973),

Pecaric (1982), however, through some modification, gives the following result

Theorem 1.3 *Let* $f : [0,1] \to \Re = (-\infty, \infty)$ *be a nonnegative and non-increasing function and let* $g : [0,1] \to \Re$ *be an integrable function such that* $0 \leq g(t) \leq 1$ *for each* $t \in [0,1]$. *If* $p \geq 1$, *then*

$$\left(\int_0^1 f(t)\,g(t)\,dt\right)^p \leq \int_0^{\mathrm{a}} f^{\,p}(t)\,dt, \tag{3}$$

where $\mathrm{a} = \left(\int_0^1 g(t)\,dt\right)^p$.

The aim of this paper is to give a generalization of Theorem 1.2, as well as other new results concerning Steffensen's inequality.

2 Results

The following theorem gives a generalization of Theorem 1.2.

Theorem 2.1 *Let* $f, h : [a,b] \to \Re$ *be nonnegative functions with f non-increasing and let* $g : [a,b] \to \Re$ *be an integrable function such that* $0 \leq L^{p-1} g(t) \leq h(t)$ *for each* $t \in [a,b]$, *where* $L = \int_a^b g(t)\,dt$. *If* $p \geq 1$, *then*

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \leq \int_a^{a+\mathrm{a}} f^{\,p}(t)\,h(t)\,dt, \tag{4}$$

where $\displaystyle\int_a^{a+\alpha} h(t)\,dt = L^p.$

Proof. Since by Hölder's inequality

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \le \left(\int_a^b f^{\,p}(t)\,g(t)\,dt\right)\left(\int_a^b g(t)\,dt\right)^{p-1},$$

then, it is sufficient to prove that

$$\int_a^{a+\alpha} f^{\,p}(t)\,h(t)\,dt \ge L^{p-1}\int_a^b f^{\,p}(t)\,g(t)\,dt.$$

We have

$$\begin{aligned}
&\int_a^{a+\alpha} f^{\,p}(t)\,h(t)\,dt - L^{p-1}\int_a^b f^{\,p}(t)\,g(t)\,dt \\
&\quad= \int_a^{a+\alpha} f^{\,p}(t)\,h(t)\,dt - L^{p-1}\int_a^{a+\alpha} f^{\,p}(t)\,g(t)\,dt - L^{p-1}\int_{a+\alpha}^{b} f^{\,p}(t)\,g(t)\,dt \\
&\quad= \int_a^{a+\alpha} f^{\,p}(t)\left(h(t) - L^{p-1}g(t)\right)dt - L^{p-1}\int_{a+\alpha}^{b} f^{\,p}(t)\,g(t)\,dt \\
&\quad\ge f^{\,p}(a+\alpha)\int_a^{a+\alpha}\left(h(t) - L^{p-1}g(t)\right)dt - L^{p-1} f^{\,p}(a+\alpha)\int_{a+\alpha}^{b} g(t)\,dt \\
&\quad= f^{\,p}(a+\alpha)\left(\int_a^{a+\alpha} h(t)\,dt - L^{p-1}\int_a^b g(t)\,dt\right) \\
&\quad= f^{\,p}(a+\alpha)\left(\int_a^{a+\alpha} h(t)\,dt - L^{P}\right) = 0.
\end{aligned}$$

As an example, we can take:

$$f(t) = \frac{1}{1+t^2}, \quad g(t) = t, \;\; h(t) = 1, \;\; p = 2, \;\; a = 0, \;\; b = 1.$$

Remark. Theorem 1.2 follows from Theorem 2.1 by putting
\

$$a = 0, \;\; b = 1, \;\; h(t) = 1.$$

Corollary 2.2. *Let* $f, h : [a,b] \to \Re$ *be nonnegative functions with f non-increasing and let* $g : [a,b] \to \Re$ *be an integrable function such that* $0 \le L^{p-1} g(t) \le h(t)$ *for each* $t \in [a,b]$, *where* $L = \int_a^b g(t)\,dt$. *If* $p \ge 1$, *then*

$$\left(\int_a^b f(t) G(t)\,dt \right)^p \le \int_a^{a+\infty} F^p(t) h(t)\,dt, \tag{5}$$

where $\int_a^{a+\infty} h(t)\,dt = L^p$, *and F, G are defined by*

$$G(t) = \int_a^t g(u)\,du, \qquad F(t) = \int_t^b f(t)\,dt .$$

In particular, if g is non-increasing, then

$$\int_0^{\infty} \left(\int_t^1 f(u)\,du \right)^p dt \ge \left(\int_0^{\infty} t f(t) g(t)\,dt \right)^p .. \tag{6}$$

Proof. We have via Theorem 2.1,

$$\int_a^{a+\infty} F^p(t) h(t)\,dt \ge \left(\int_a^b F(t) g(t)\,dt \right)^p$$

$$= \left(\int_a^b \int_t^b f(u) g(t)\,du dt \right)^p$$

$$= \left(\int_a^b f(u) \int_a^t g(t)\,dt du \right)^p$$

$$= \left(\int_a^b f(t) G(t)\,dt \right)^p .$$

Now, if $a = 0,\ b = 1$ in (4), we have

$$\int_0^{\infty} \left(\int_t^1 f(u)\,du \right)^p h(t)\,dt \ge \left(\int_0^1 f(t) G(t)\,dt \right)^p$$

$$= \left(\int_0^1 f(t) \int_0^t g(u)\,du dt \right)^p$$

$$\geq \left(\int_0^1 tf(t)\,g(t)\,dt \right)^p .$$

The following gives the inequality for the case $0 < p \leq 1$.

Theorem 2.3. *Let* $f, h : [a,b] \to \Re$ *be nonnegative functions and f non-decreasing and let* $g : [a,b] \to \Re$ *be an integrable function such that* $0 \leq L^{p-1} g(t) \leq h(t)$ *for each* $j\, t \in [a,b]$, *where* $L = \int_a^b g(t)\,dt$. *If* $0 < p \leq 1$, *then*

$$\left(\int_a^b f(t)\,g(t)\,dt \right)^p \geq \int_a^{a+\alpha} f^p(t)\,h(t)\,dt, \tag{7}$$

where $\int_a^{a+\alpha} h(t)\,dt = L^p$.

Proof. Since by Hölder's inequality

$$\left(\int_a^b f(t)\,g(t)\,dt \right)^p \geq \left(\int_a^b f^p(t)\,g(t)\,dt \right) \left(\int_a^b g(t)\,dt \right)^{p-1},$$

then, it is sufficient to prove that

$$\int_a^{a+\alpha} f^p(t)\,h(t)\,dt \leq L^{p-1} \int_a^b f^p(t)\,g(t)\,dt.$$

We have

$$L^{p-1} \int_a^b f^p(t)\,g(t)\,dt - \int_a^{a+\alpha} f^p(t)\,h(t)\,dt$$

$$= L^{p-1} \int_a^{a+\alpha} f^p(t)\,g(t)\,dt + L^{p-1} \int_{a+\alpha}^b f^p(t)\,g(t)\,dt - \int_a^{a+\alpha} f^p(t)\,h(t)\,dt$$

$$= \int_a^{a+\alpha} f^p(t) \left(L^{p-1} g(t) - h(t) \right) dt + L^{p-1} \int_{a+\alpha}^b f^p(t)\,g(t)\,dt$$

$$\geq f^p(a+\alpha) \int_a^{a+\alpha} \left(L^{p-1} g(t) - h(t) \right) dt + L^{p-1} f^p(a+\alpha) \int_{a+\alpha}^b g(t)\,dt$$

$$= f^p(a+\alpha) \left(L^{p-1} \int_a^b g(t)\,dt - \int_a^{a+\alpha} h(t)\,dt \right)$$

$$= f^{p}(a+\alpha)\left(L^{p} - \int_{a}^{a+\alpha} h(t)\,dt \right) = 0.$$

4 Steffensen's Inequality via Double Integrals

Theorem 3.1 *Let* $f,h:[a,b]\times[c,d]\to\Re$ *be nonnegative functions and f non-increasing w.r.t s and let* $g:[a,b]\times[c,d]\to\Re$ *be an integrable function such that*

$0 \le L^{p-1} g(s,t) \le h(s,t)$ *for all* $s\in[a,b], t\in[c,d]$, *where* $L=\int_{a}^{b} g(s,t)\,ds$. *If* $p\ge 1$,

then

$$\frac{1}{(b-a)^{p}}\left(\int_{c}^{d}\int_{a}^{b} f(s,t)\,g(s,t)\,dsdt \right)^{p} \le \int_{c}^{d}\int_{a}^{a+\alpha} f^{p}(s,t)\,h(s,t)\,dsdt, \tag{8}$$

where $\int_{a}^{a+\alpha} h(s,t)\,ds = L^{p},\ 0 \le \alpha \le b-a$. *In particular*

$$\frac{1}{(b-a)^{p}}\left(\int_{a}^{b}\int_{a}^{b} f(s,t)\,g(s,t)\,dsdt \right)^{p} \le \int_{a}^{b}\int_{a}^{a+\alpha} f^{p}(s,t)\,h(s,t)\,dsdt, \tag{9}$$

$$\frac{1}{(b-a)^{p}}\left(\int_{a}^{a+\alpha}\int_{a}^{b} f(s,t)\,g(s,t)\,dsdt \right)^{p} \le \int_{a}^{a+\alpha}\int_{a}^{a+\alpha} f^{p}(s,t)\,h(s,t)\,dsdt,$$

(10)

Proof. Since for any non-negative function $m(x)$, we have, by Hölder's inequality

$$\left(\int_{a}^{b} m(x)\,dx \right)^{p} \le \int_{a}^{b} m^{p}(x)\,dx \left(\int_{a}^{b} dx \right)^{p-1} = (b-a)^{p-1}\int_{a}^{b} m^{p}(x)\,dx,$$

then, we have via above and Theorem 2.1,

$$\left(\int_{a}^{b} f(s,t)\,g(s,t)\,ds \right)^{p} \le \int_{a}^{a+\alpha} f^{p}(s,t)\,h(s,t)\,ds,$$

which implies

$$\int_{c}^{d}\left(\int_{a}^{b} f(s,t)\,g(s,t)\,ds \right)^{p} dt \le \int_{c}^{d}\int_{a}^{a+\alpha} f^{p}(s,t)\,h(s,t)\,dsdt.$$

But

$$\int_c^d \int_a^{a+\alpha} f^p(s,t)\, g(s,t)\, dsdt \ge \frac{1}{(b-a)^p}\left(\int_c^d \int_a^b f(s,t)\, h(s,t)\, dsdt \right)^p ,$$

then

$$\frac{1}{(b-a)^p}\left(\int_c^d \int_a^b f(s,t)\, g(s,t)\, dsdt \right)^p \le \int_c^d \int_a^{a+\alpha} f^p(s,t)\, h(s,t)\, dsdt\, .$$

Theorem 3.2 *Let* $f, h : [a,b]\times[c,d] \to \Re$ *be nonnegative functions and f non-increasing w.r.t s and t, let* $g : [a,b]\times[c,d] \to \Re$ *be an integrable function such that*

$0 \le M^{2-1/p} g(s,t) \le h(s,t)$ *for all* $s \in [a,b], t \in [c,d]$, *where* $M^p = \int_a^b g(s,t)\, ds$.

If $p \ge 1/2$, *then*

$$\left(\int_a^{a+\alpha} \int_a^b f(s,t)\, g(s,t)\, dsdt \right)^{2p} \le \int_a^{a+\alpha} \int_a^{a+\alpha} f^{2p}(s,t)\, h(s,t)\, dsdt \qquad (11)$$

where $\int_a^{a+\alpha} \int_a^{a+\alpha} h(s,t)\, dsdt = M^{2-1/p} \int_a^{a+\alpha} \int_a^b g(s,t)\, dsdt.$

Proof.

$$\int_a^{a+\alpha} \int_a^{a+\alpha} f^{2p}(s,t)\, h(s,t)\, dsdt - M^{2-1/p} \int_a^{a+\alpha} \int_a^b f^{2p}(s,t)\, g(s,t)\, dsdt$$

$$= \int_a^{a+\alpha} \int_a^{a+\alpha} f^{2p}(s,t)\, h(s,t)\, dsdt - M^{2-1/p} \int_a^{a+\alpha} \int_a^{a+\alpha} f^{2p}(s,t)\, g(s,t)\, dsdt$$

$$- M^{2-1/p} \int_a^{a+\alpha} \int_{a+\alpha}^b f^{2p}(s,t)\, g(s,t)\, dsdt$$

$$= \int_a^{a+\alpha} \left(\int_a^{a+\alpha} f^{2p}(s,t)\, h(s,t)\, ds - M^{2-1/p} \int_a^{a+\alpha} f(s,t)\, g(s,t)\, ds \right.$$

$$\left. - M^{2-1/p} \int_{a+\alpha}^b f^{2p}(s,t)\, g(s,t)\, ds \right) dt$$

$$= \int_a^{a+\alpha} \left(\int_a^{a+\alpha} f^{2p}(s,t)\big(h(s,t) - M^{2-1/p} g(s,t)\big) ds - M^{2-1/p} \int_{a+\alpha}^{b} f^{2p}(s,t)\, g(s,t)\, ds \right) dt$$

$$\geq \int_a^{a+\alpha} \left(f^{2p}(a+\alpha, t) \left(\int_a^{a+\alpha} h(s,t)\, ds - M^{2-1/p} \int_a^{a+\alpha} g(s,t)\, ds \right) - \right.$$

$$\left. - M^{2-1/p} f^{2p}(a+\alpha, t) \int_{a+\alpha}^{b} g(s,t)\, ds \right) dt$$

$$= \int_a^{a+\alpha} \left(f^{2p}(a+\alpha, t) \left(\int_a^{a+\alpha} h(s,t)\, ds - M^{2-1/p} \int_a^{b} g(s,t)\, ds \right) dt \right.$$

$$\geq f(a+\alpha, a+\alpha) \left(\int_a^{a+\alpha} \int_a^{a+\alpha} h(s,t)\, ds dt - M^{2-1/p} \int_a^{a+\alpha} \int_a^{b} g(s,t)\, ds dt \right)$$

$$\geq 0.$$

5 Steffensen's Inequality via Hölder's, Minkowski's and Hardy-Hilbert's Inequalities

Theorem 4.1. *Let the conditions of Theorem 2.1 are satisfied. Let* $p > 1$, $1/p + 1/q = 1$. *Then*

$$\int_a^b f_1^{\frac{1}{p}}(t)\, f_2^{\frac{1}{q}}(t)\, g(t)\, dt \leq \left(\int_a^{a+\alpha} f_1^{p}(t)\, h(t)\, dt \right)^{1/p^2} \left(\int_a^{a+\alpha} f_2^{q}(t)\, h(t)\, dt \right)^{1/q^2}. \qquad 12)$$

Proof. We have, by Theorem 2.1, via Hölder's inequality

$$\int_a^b f_1^{\frac{1}{p}}(t)\, f_2^{\frac{1}{q}}(t)\, g(t)\, dt$$

$$= \int_a^b f_1^{\frac{1}{p}}(t)\, g^{\frac{1}{p}}(t)\, f_2^{\frac{1}{q}}(t)\; g^{\frac{1}{q}}(t)\, dt$$

$$\leq \left(\int_a^b f_1(t)\, g(t)\, dt \right)^{1/p} \left(\int_a^b f_2(t)\, g(t)\, dt \right)^{1/q}$$

$$\leq \left(\int_a^{a+\alpha} f_1^p(t)\, h(t)\, dt \right)^{1/p^2} \left(\int_a^{a+\alpha} f_2^q(t)\, h(t)\, dt \right)^{1/q^2} .$$

Theorem 4.2. . *Let the conditions of Theorem 2.1 are satisfied. Let* $p > 1,$ $1/p + 1/q = 1,\ r \geq 1.$ *Then*

$$\left(\int_a^b f_1(t)\, f_2(t)\, g^2(t)\, dt \right)^{1/r} \leq \frac{1}{p} \left(\int_a^{a+\alpha} f_1^p(t)\, h(t)\, dt \right)^{1/r} + \frac{1}{q} \left(\int_a^{a+\alpha} f_2^q(t)\, h(t)\, dt \right)^{1/r} .$$

(13)

Proof. We have, via Minkowski's inequality

$$\left(\int_a^b f_1(t)\, f_2(t)\, g^2(t)\, dt \right)^{1/r} = \left(\int_a^b \left(f_1^{\frac{1}{r}}(t)\, g^{\frac{1}{r}}(t)\, f_2^{\frac{1}{r}}(t)\, g^{\frac{1}{r}}(t) \right)^r dt \right)^{1/r}$$

$$\leq \left(\int_a^b \left(\frac{f_1^{\frac{p}{r}}(t)\, g^{\frac{p}{r}}(t)}{p} + \frac{f_2^{\frac{q}{r}}(t)\, g^{\frac{q}{r}}(t)}{q} \right)^r dt \right)^{1/r}$$

$$\leq \left(\int_a^b \frac{f_1(t)\, g(t)}{p^r} dt \right)^{1/r} + \left(\int_a^b \frac{f_2(t)\, g(t)}{q^r} dt \right)^{1/r}$$

$$= \frac{1}{p} \left(\int_a^b f_1(t)\, g(t)\, dt \right)^{1/r} + \frac{1}{q} \left(\int_a^b f_2(t)\, g(t)\, dt \right)^{1/r}$$

$$\leq \frac{1}{p} \left(\int_a^{a+\alpha} f_1^p(t)\, h(t)\, dt \right)^{1/r} + \frac{1}{q} \left(\int_a^{a+\alpha} f_2^q(t)\, h(t)\, dt \right)^{1/r} .$$

Hardy-Hilbert's integral inequality has been extended by Yang (2001) via presenting the following result

If $\lambda > 2 - \min\{p, q\},\ 0 < \int_0^{\infty} x^{1-\lambda} f^p(x)\, dx < \infty,\ 0 < \int_0^{\infty} x^{1-\lambda} g^q(x)\, dx < \infty,\ p > 1,$ $1/p + 1/q = 1.$ *Then*

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)\, g(y)}{(x+y)^{\lambda}} dxdy < k_{\lambda}(p) \left(\int_0^{\infty} x^{1-\lambda} f^p(x)\, dx \right)^{1/p} \left(\int_0^{\infty} x^{1-\lambda} g^q(x)\, dx \right)^{1/q} ,$$

where $k_{\text{¢}}(p) = B\left(\frac{p+\text{¢}-2}{p}, \frac{q+\text{¢}-2}{q}\right),$ *is the best possible, B stands for the beta function.*

Theorem 4.3 *Let the conditions of Theorem 2.1 are satisfied for both f and g with* $x^{1-\text{¢}} f^{p}(x)$, $x^{1-\text{¢}} g^{q}(x)$, $h(t)=1,\ a=0,\ b=\infty.$ *Let* $p>1,\ \ 1/p+1/q=1,\ \ \text{¢} > 2-\min\{p,q\}$. *Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)h(y)\,g^{\frac{1}{p}}(x)\,g^{\frac{1}{q}}(y)}{(x+y)^{\text{¢}}}\,dxdy$$

$$\le k_{\text{¢}}(p)\left(\int_0^{a} x^{(1-\text{¢})p} f^{p^2}(x)\ dx\right)^{1/p^2}\left(\int_0^{a} x^{(1-\text{¢})q} h^{q^2}(x)\ dx\right)^{1/q^2} \tag{14}$$

Proof. We have via Hardy-Hilbert's integral inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)h(y)\,g^{\frac{1}{p}}(x)\,g^{\frac{1}{q}}(y)}{(x+y)^{\text{¢}}}\,dxdy$$

$$< k_{\text{¢}}(p)\left(\int_0^\infty x^{1-\text{¢}} f^{p}(x)\,g(x)\,dx\right)^{1/p}\left(\int_0^\infty x^{1-\text{¢}} h^{q}(x)\,g(x)\,dx\right)^{1/q}$$

$$\le k_{\text{¢}}(p)\left(\int_0^{a} x^{(1-\text{¢})p} f^{p^2}(x)\ dx\right)^{1/p^2}\left(\int_0^{a} x^{(1-\text{¢})q} h^{q^2}(x)\ dx\right)^{1/q^2}$$

6 Conclusion

Three results are given as generalizations of Steffensen's inequality. Other two results concerning inequalities are similar to Steffensen's inequality but with double integrals. As well three other results are similar to Steffensen's inequalities via Hölder's, Minkowski's and Hardy-Hilbert's inequalities.

Acknowledgements

The author is so grateful to the referees who read through the paper very well and corrected many mistakes.

Competing Interests

Author has declared that no competing interests exist.

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