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On Spanning Tree Packing Number of the Complement of Generalized Petersen Graph and Cocktail Party Graph

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

For any graph G, the spanning tree packing number of $\sigma(G)$, is the maximum number of edge-disjoint spanning trees contained in G. In this study, we determined the maximum number of edge-disjoint spanning trees of the generalized petersen graph and cocktail graph.

Keywords: Spanning tree number; petersen graph; edge-disjoint.

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1 Introduction

The spanning-tree packing number, denoted as $\sigma(G)$, of a graph G with n vertices represents the maximum count of edge-disjoint spanning trees present in G. This metric is widely used to assess the reliability of communication networks and has been extensively studied by researchers. For further exploration of this topic, comprehensive surveys conducted by Palmer [1] and Ozeki and Yamashita [2] provide valuable insights. The determination of the maximum number of edge-disjoint spanning trees in a given graph G can be achieved in polynomial time, as discussed in [3].

Peng and Tay [4] conducted a study focusing on the spanning-tree packing numbers of Cartesian products formed by combining different combinations of complete graphs, cycles, and complete multipartite graphs. Subsequently, Ku, Wang, and Hung [5] established the following result: for two connected graphs G and H, the spanning-tree packing number of their Cartesian product is equal to or greater than the sum of the spanning-tree packing numbers of G and H, minus one. In [6], Li, H. et al. obtained a sharp lower bound for the spanning-tree packing number of lexicographic product graphs.

In this paper, we determine the spanning tree packing number of the complement of the Generalized Petersen Graphs, $\overline{G(n,k)}$, and the Cocktail party graphs, CP_n.

2 Preliminary Notes

This section contains some of the fundamental concepts necessary for the understanding of the study. Definitions that are not in this paper can be found on [7], [8], [9].

Definition 2.1. [10] The complement $\overline{G}$ of a graph G is the grpah with vertex set $V(G)$ such that two verticesare adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G.

Definition 2.2. [10] An edge-connectivity $\lambda(G)$ of a graph g is the minimum number of edges in graph G whose deletion disconnects the graph.

Theorem 2.1. *[11] The edge connectivity of G satisfies $\lambda(G) \geq 2k$ if and only if for any set E_k of k edges of G, then the subgraph $H = G - E_k$ has an edge disjoint trees.*

Corollary 2.2. *[11] If $\lambda(G) \geq 2k$ then G has k edge-disjoint spanning trees. The lower bound is*

$$\left\lfloor \frac{\lambda(G)}{2} \right\rfloor \leq \sigma(G),$$

where the upper bound is

$$\sigma(G) \leq \left\lfloor \frac{|E(G)|}{|V(G) - 1|} \right\rfloor.$$

Remark 2.1. [12] For any complement of generalized petersen graph $\overline{G(n,k)}$ with $n \geq 3$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $|E(\overline{G(n.k)})| = 2n(n-2)$ and $deg_{\overline{G(n.k)}}(v) = 2(n-2)$ for all $v \in V(\overline{G(n.k)})$.

Remark 2.2. [13] For any Cocktail Party graph CP_n with $n \geq 2$,

$$|E(CP_n)| = 2n(n-1), \text{ and } deg_{CP_n}(v) = 2(n-1) \text{ for all } v \in V(CP_n)$$

Remark 2.3. [10] For any regular graph G, $\delta(G) = deg_G(v)$.

Theorem 2.3. *[14] Let G be a graph with order p, minimum degree $\delta(G)$, and edge connectivity $\lambda(G)$. If $\lambda(G) \geq \frac{1}{2}p$, then $\lambda(G) = \delta(G)$.*

Remark 2.4. [1] For any connected cubic graph G for which $|V(G)| \geq 6$, $\sigma(G) = 1$.

Definition 2.3. [10] A vertex-connectivity $\kappa(G)$ of a graph G is the minimum number of nodes in a graph G whose deletion disconnects G.

Definition 2.4. [10] A graph H is subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a sapnning subgraph of G.

Definition 2.5. [1] For any graph G, the spanning tree packing number (STP) denoted by $\sigma(G)$, is the maximum number of edge-disjoint spanning trees contained in G.

Definition 2.6. [12] Generalized petersen graph is connected cubic graph consisting of an inner star polygon (n, k) and an outer regular polygon (n) with correspoding vertices in the inner and outer polygons connected by the edges where $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

Definition 2.7. [15] Ladder rung graph is graph of order n denoted by nP_2 the graph union of n copies of path graph P_2.

Definition 2.8. [13] Cocktail party graph (CP_n) is a grpah consisting of two rows of paired nodes in which all nodes except the paired ones are connected with straight lines. It is the graph complement of the ladder rung graph nP_2.

3 Main Results

Remark 3.1. For any Generalized petersen graph $G(n, k)$ for $n \geq 3$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $\sigma(G(n, k)) = 1$.

Theorem 3.1. *For a generalized petersen graph $G(n, k)$, $\sigma\overline{G(n, k)} = n - 2$ for all $n \geq 3$.*

Proof: We consider the following cases:

Case 1: $n - 2 \leq \sigma(\overline{G(n, k)})$.

By Corollary 2.2,

$$\left\lfloor \frac{\lambda(G)}{2} \right\rfloor \leq \sigma(G)$$

Now, by Remark 2.1, $deg_{\overline{(n,k)}}(v) = 2(n - 2)$, for all $v \in V(\overline{(G(n, k))})$. Since $\overline{G(n, k)}$ is also a regular graph, it follows that $\delta(\overline{G(n, k)}) = \Delta(\overline{G(n, k)}) = deg_{\overline{G(n,k)}}(v) = 2(n - 2)$, $\lambda(\overline{G(n, k)}) = \delta(\overline{G(n, k)})$. By Theorem 2.3

$$\begin{aligned} \delta(\overline{G(n, k)}) &\geq \frac{|V(\overline{G(n, k)})|}{2} \\ 2(n - 2) &\geq \frac{2n}{2} \\ 2n - 4 &\geq n \\ n &\geq 4. \end{aligned}$$

This is true since $|V\overline{G(n, k)})| \geq 6$. Hence $\lambda\overline{G(n, k)}) = 2(n - 2)$. Now we have

$$\sigma\overline{G(n, k)} \geq \left\lfloor \frac{\lambda(\overline{G(n, k)})}{2} \right\rfloor = \left\lfloor \frac{2(n - 2)}{2} \right\rfloor = n - 2.$$

Case 2: $\sigma\overline{G(n, k)} \leq n - 2$

On the other hand, by 2.2, and Remark 2.1, we get

$$\sigma\overline{G(n,k)} \leq \left\lfloor \frac{|E((\overline{G(n,k)})|}{|V(\overline{G(n,k)}| - 1} \right\rfloor$$
$$= \left\lfloor \frac{2n(n-2)}{2n-1} \right\rfloor$$

Now, to show that $\left\lfloor \frac{2n(n-2)}{2n-1} \right\rfloor = n-2$ *fro* $n \geq 3$ *we use Remark* **??***, i.e., we need to show that* $n-2 \leq \left\lfloor \frac{2n(n-2)}{2n-1} \leq (n-2)+1$*. Consider the following claims:*

Claim 1: $n-2 \leq \frac{2n(n-2)}{2n-1}$
Verifying algebraically,

$$(n-2)(2n-1) \leq 2n(n-2)$$
$$2n^2 - 2 - 4n + 2 \leq 2n^2 - 4n$$
$$-n + 2 \leq 0$$
$$-n \leq -2.$$

This shows that Claim 1 is true since $n \geq 3$ *by assumption.*

Claim 2: $\frac{2n(n-2)}{2n-1} \leq (n-2)+1$
Verifying algebraically,

$$2n(n-2) \leq n - 1(2n-1)$$
$$2n^2 - 4n \leq 2n^2 - 2n - n + 1$$
$$-4n \leq -3n + 1$$
$$-n \leq -1$$

This shows that claim 2 is true since $n \geq 3$ *by assumption. Combining by claim 1 and claim 2, we get*

$$\sigma(\overline{G(n,k)}) \leq \left\lfloor \frac{2(n-2)}{2} \right\rfloor = n-2.$$

By case 1 and case2, this implies that

$$n-2 \leq \sigma(\overline{G(n,k)}) \leq n-2.$$

Therefore, $\sigma(\overline{G(n,k)}) = n-2$. □

The following are consequence of Remark 3.1, and Theorem 3.1.

Remark 3.2. The absolute difference between $\sigma(\overline{G(n,k)})$ and $\sigma(G(n,k))$ is increasing.

The next results shows the STP od Cocktail party graph denoted by CP_n.

Theorem 3.2. *For a Cocktail party graph denoted by* (CP_n) *with* $n \geq 2$*,*

$$\sigma(CP_n) = n-1.$$

Proof:

We consider the following cases:

Case 1: $n-1 \leq \sigma(CP_n)$

By Corollary 2.2,

$$\frac{\lambda(G)}{2} \leq \sigma(G).$$

Now, by Remark 2.2, $deg_{CP_n}(v) = 2(n-1)$ *for all* $v \in V(CP_n)$*. Since* (CP_n) *is also a regular graph, it follows that* $\delta(CP_n) = \Delta(CP_n) = deg_{CP_n} = 2(n-1)$*. Now to show that* $\sigma(CP_n) \geq \frac{|V(CP_n)|}{2}$ *so that* $\lambda(CP_n) = \sigma(CP_n)$*. By Theorem 2.3*

$$\begin{aligned} 2(n-1) &\geq \frac{2n}{2} \\ 2n-2 &\geq n \\ n-2 &\geq 0 \\ n &\geq 2. \end{aligned}$$

Hence, $\lambda(CP_n) = deg_{CP_n}(v) = 2(n-1)$*. Now, by Corollary 2.2, we have*

$$\begin{aligned} \sigma(G(n,k)) &\geq \left\lfloor \frac{2(n-1)}{2} \right\rfloor \\ \sigma(G(n,k)) &= n-1. \end{aligned}$$

Case 2: $\sigma(CP_n) \leq n-1$

On the other hand by Corollary 2.2 and Remark 2.2

$$\begin{aligned} \sigma(\overline{CP_n}) &\leq \left\lfloor \frac{|E(\overline{CP_n})|}{|V(\overline{CP_n})|-1} \right\rfloor \\ &= \left\lfloor \frac{2n(n-2)}{2n-1} \right\rfloor. \end{aligned}$$

To show that $\left\lfloor \frac{2n(n-2)}{2n-1} \right\rfloor = n-1$ *for* $n \geq 2$*, we used Remark 2.4,i.e., we need to show that* $n-1 \leq \frac{2n(n-2)}{2n-1} \leq (n-1)+1$*. Consider the follwing Claims below:*

Claim 1: $n-1 \leq \frac{2n(n-2)}{2n-1}$.
Verifying algebraically,

$$\begin{aligned} (n-1)(2n-1) &\leq 2n(n-1) \\ 2n^2-n-2n+1 &\leq 2n^2-2n \\ -n+1 &\leq 0 \\ -n &\leq -1. \end{aligned}$$

Claim 2: $\frac{2n(n-2)}{2n-1} \leq (n-1)+1$.
Verifying algebraically,

$$\begin{aligned} 2n(n-1) &\leq n(2n-1) \\ 2n^2-2n &\leq 2n^2-2 \\ -2n &\leq -n \\ -n &\leq 0 \end{aligned}$$

This shows that Claim 2 is true since $n \geq 2$ *by assumption. Thus*

$$\sigma(CP_n) \leq \left\lfloor \frac{2n(n-1)}{2n-1} \right\rfloor = n-1.$$

By case 1 and case 2 this implies that,

$$n-1 \leq \sigma(CP_n) \leq n-1.$$

Therefore,

$$\sigma(CP_n) = n-1$$

□

Remark 3.3. The spanning tree packing number of $\overline{CP_n}$ is 0.

Remark 3.4. The difference between $\sigma(CP_n)$ and $\sigma(\overline{CP_n})$ is increasing.

4 Conclusion

In this paper, we have successfully determined the spanning tree packing numbers for two specific graph families: the complement of Generalized Petersen Graphs, denoted as $\overline{G(n,k)}$, and the Cocktail party graphs, denoted as CP_n. By analyzing these graph structures, we have obtained valuable insights into the properties and characteristics of their spanning tree packing numbers.

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Competing Interests

The authors declare that they have no competing interests.

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