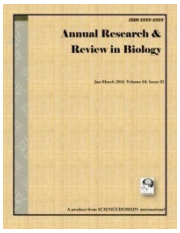

Annual Research & Review in Biology
4(3): 560-576, 2014

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Optimal Control of an Epidemic Model of Leptospirosis with Nonlinear Saturated Incidences

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Authors' contributions

This work was carried out in collaboration between all authors. Author SFS designed the model, performed the basic of the model, and wrote the first draft of the manuscript. Authors MAK and SI managed the analyses of the study. Authors GZ, IHJ and SAK managed the literature searches. All authors read and approved the final manuscript.

Original Research Article

Received 13th August 2013
Accepted 15th September 2013
Published 7th November 2013

ABSTRACT

In this paper, we consider a leptospirosis epidemic model with non-linear saturated incidence by applying the optimal control techniques to eradicate the infection in the human population. Leptospirosis is the disease which effects human as well as Cattle. Our aim is to find such optimal control techniques for the eradication of leptospira in the host population. To do this, we define three control variables, one for human and the second and third one for Vector population. First we find the existence of the control problem and then we characterize the optimal control problem by using the well-known method of Pontryagin's Maximum Principle. The numerical simulations of both the system are solving

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by using backward Runge-kutta order four schemes. Finally, the numerical results of both the systems are presented for comparison.

Keywords: Leptospirosis; Pontryagin's maximum principle; optimal control; numerical simulations.

1. INTRODUCTION

Mathematical models have played an important role in understanding the epidemiology of the infectious disease [1,2]. The model provides quantitative descriptions of the complicated, nonlinear process of disease transmission and help us to obtain inside into the dynamics of the disease and we are able to make such decision for public health policy. Many mathematical models [3,4,5] have been proposed to represents the compartmental dynamics of both susceptible, infected and recovered human and vector population .Triampo et al. [6] considered a deterministic model for the transmission of leptospirosis disease presented in [6]. In their work they considered a number of leptospirosis disease in Thailand and shown the numerical simulations. Zaman et al. [7] using the real data of Thailand for their numerical simulations, find the global stability of both the Disease-free and endemic equilibrium and also the backward bifurcation for different set of parameters. Zaman [8] considered the real data presented in [5] to study the dynamical behavior and role of optimal control theory of this disease. Pongsumpun et al. [9] developed mathematical models to study the behavior of leptospirosis disease. In their work, they represent the rate of change for both rats and human population. The human population is further divided into two main groups Juveniles and adults. A variety of nonlinear incidence rate have been used in epidemic models [10,11,12]. Liu et al. [13] demonstrated that in cases where the host could exhibit lasting immunity to infection that nonlinear incidence rates of infection could greatly expand the breadth of dynamics caused by the disease. By investigating cholera epidemic spread in Bari in 1973, Capasso and Serio [12] consider a saturated incidence in his model.

In this paper, we consider the basic model studied in [8] to incorporate some important epidemiological features. We use optimal control theory to reduce the proportion of the infected human and infected vector population by using multiple controls. At the long-term level of infected human, every infected human on average causes one further secondary cases. Therefore, if we can reduce the number of infected human further, so the disease does well and increase the recovered human. Here we define the control variables, the first control is cover all cuts, wear dry, full-cover boots, shoes and long sleeve shirts when handling animals. The second control represents wash hands thoroughly on a regular basis and shower after work. To do this, we first show the existence of theoptimal control system. Then, by solving the optimality system numerically, which consists of the original state system, the adjoint system and their boundary conditions by using the real data presented for leptospirosis epidemic in Thailand. We also conclude by discussing results of the numerical simulations for our epidemic mathematical model.

The paper is organized as follows. Section 2 is devoted to the mathematical formulation of Leptospirosis disease. We discuss the existence of the control problem in their discussion in Section 3. Existence of the control problem is discussed in Section 4. In Section 5 we discuss the numerical results of the control problem and finally we wind up our work with the conclusion in Section 6.

2. MATERIAL AND METHODS

In this section, we consider the epidemic model of leptospirosis [8], with the nonlinear saturated incidences rates. First, we formulate the model in detail and define the parameter involve in the model. To do this, we assume that $S_h(t)$ represents a number of susceptible human at time "t"; $I_h(t)$ represents a number of human in the population, which is infected from the leptospirosis disease at time "t"; $R_h(t)$ represents number of human in the population which is recovered from the infection at time ; we denote the total population size by N_h, with $N_h(t) = S_h(t) + I_h(t) + R_h(t)$. For vector population, let $S_v(t)$ are susceptible vector and $I_v(t)$ are infectious vector at time "t". The total population size of vector population is denoted by $N_v(t)$ with $N_v(t) = S_v(t) + I_v(t)$. By the interaction of both human and vector population, we get the following system of five differential equations is given by:

$$\begin{aligned}
\frac{dS_h}{dt} &= b_h - \mu_h S_h - S_h\left(\frac{\beta_2 I_v}{1+\alpha_1 I_v} + \frac{\beta_1 I_h}{1+\alpha_2 I_h}\right) + \lambda_h R_h,\\
\frac{dI_h}{dt} &= S_h\left(\frac{\beta_2 I_v}{1+\alpha_1 I_v} + \frac{\beta_1 I_h}{1+\alpha_2 I_h}\right) - (\mu_h + \delta_h + \gamma_h) I_h,\\
\frac{dR_h}{dt} &= \gamma_h I_h - (\mu_h + \lambda_h) R_h,\\
\frac{dS_v}{dt} &= b_v - \gamma_v S_v - \frac{\beta_3 S_v I_h}{1+\alpha_2 I_h},\\
\frac{dI_v}{dt} &= \frac{\beta_3 S_v I_h}{1+\alpha_2 I_h} - \gamma_v I_v - \delta_v I_v,
\end{aligned} \tag{1}$$

with the initial conditions,

$$S_h(0) \geq 0, \qquad I_h(0) \geq 0, \qquad R_h(0) \geq 0, \qquad S_v(0) \geq 0 \qquad I_v(0) \geq 0. \tag{2}$$

Here b_h is the recruitment rate of the human population, $\beta_1, \beta_2, \beta_3$ respectively represent the transmission coefficient between human, susceptible human and infected vector and susceptible vector and infected human. Natural mortality rate of human population is represented by μ_h. λ_h is constant of proportionality where the infected human become susceptible again. Disease death rate for human population is denoted by δ_h. Natural mortality rate of vector population is shown by γ_v. δ_v is the disease death rate of vector. α_1 The parameter measure the inhibitory effect of human vector population and α_2 the parameter measure the inhibitory effect of human population. b_v is the recruitment rate for vector population.

3. OPTIMAL CONTROL PROBLEM

Optimal control theory is a powerful mathematical tool which makes the decision involving complex dynamical systems [14]. Optimal control method have been used, to study the dynamics of the disease we refer the reader to [15,16], for more work[17-23], no such method is used according to the author's knowledge, to determine optimal control measure for vector-host epidemic direct transmission. The problem is to minimize the infected human and vector population and to maximize the recovered human population.

In the system (3) below we have five state variables $S_h(t), I_h(t), R_h(t), S_v(t)$ and $I_v(t)$ with three control variables u_1, u_2, u_3 .The control problem consists of a non-linear system of five differential equations is given by,

$$\begin{aligned}
\frac{dS_h}{dt} &= b_h - \mu_h S_h - S_h\Big(\frac{\beta_2 I_v(1-u_1(t))}{1+\alpha_1 I_v} + \frac{\beta_1 I_h(1-u_2(t))}{1+\alpha_2 I_h}\Big) + \lambda_h R_h,\\
\frac{dI_h}{dt} &= S_h\Big(\frac{\beta_2 I_v(1-u_1(t))}{1+\alpha_1 I_v} + \frac{\beta_1 I_h(1-u_2(t))}{1+\alpha_2 I_h}\Big) - (\mu_h + \delta_h + \gamma_h) I_h,\\
\frac{dR_h}{dt} &= \gamma_h I_h - (\mu_h + \lambda_h) R_h + (u_1(t) + u_2(t)) S_h,\\
\frac{dS_v}{dt} &= b_v - \gamma_v S_v - \frac{\beta_3 S_v I_h}{1+\alpha_2 I_h} - u_3(t)\epsilon_1 S_v,\\
\frac{dI_v}{dt} &= \frac{\beta_3 S_v I_h}{1+\alpha_2 I_h} - \gamma_v I_v - \delta_v I_v - u_3(t)\epsilon_2 I_v.
\end{aligned} \tag{3}$$

Subject to the initial conditions (2). Here we define the control variables u_1, u_2, u_3, to represent the following.

- The first control $u_1(t)$ represents that human should cover all cuts, abrasions with waterproof dressing, grazes, wear dry clothes, wearing full cover shoes, gloves and using the shirts with long sleeve when we handling the animals.
- The second control $u_2(t)$ shows that after work human should bath or shower regularly and adopt the habit to wash our hands regularly.
- Our third control $u_3(t)$ represents that to clean the home and working area.

Now we define the objective functional for the control problem to minimize the infected human population, susceptible vector and infected vector population and maximize the population of recovered human and susceptible human individuals. For this we use three control variables which is defined above. The objective functional for the control problem is given by,

$$J(u_1, u_2, u_3) = \min \int_0^T \Big(Q_o S_h + Q_1 I_h + Q_2 S_v + Q_3 I_v + \frac{1}{2}(k_o u_1^2 + k_1 u_2^2(t) + k_2 u_3^2(t))\Big) \tag{4}$$

Here the term $Q_0, Q_1, Q_2, Q_3, k_o, k_1$ and k_2 are the weight constants to keep the balance in the objective functional and also m is the constant at which the vectors are eradicated. The objective functional also contains the susceptible class and recovered class of human individuals, our goal is to maximize the recovered and susceptible human population and minimize the vector population and infection in the human population. Here we define the control set,

$$K = \left(u_i, i = 1, 2. \quad is\, lebesgue\, measureable, 0 \le u_i(t) \le 1., i = 1, to\, 3,\ t \in [0, T]\right) \tag{5}$$

The lagrangian for the control problem is defined as

$$L = \left(Q_o S_h + Q_1 I_h + Q_2 S_v + Q_3 I_v + \frac{1}{2}(k_o u_1^2 + k_1 u_2^2(t) + k_2 u_3^2(t))\right) \tag{6}$$

And the Hamiltonian represented by M is defined as

$$M = \left(Q_o S_h + Q_1 I_h + Q_2 S_v + Q_3 I_v + \frac{1}{2}(k_o u_1^2 + k_1 u_2^2(t) + k_2 u_3^2(t))\right) + \lambda_1 \frac{dS_h}{dt} + \lambda_2 \frac{dI_h}{dt} + \lambda_3 \frac{dR_h}{dt} + \lambda_4 \frac{dS_v}{dt} + \lambda_5 \frac{dI_v}{dt} \tag{7}$$

Now, we show the existence of the of the control problem (3), in the following section by using the leipstics conditions.

4. EXISTENCE OF THE CONTROL PROBLEM

To show the existence of the control problem (3), we write the system (3) in the following form, and S_h, I_h, R_h, S_v and I_v respectively represent the state variables susceptible human, infected human, recovered human, susceptible vector and infected vector. The control variables are u_1, u_3 and u_3.

Let

$$Z' = BZ + F(Z), \tag{8}$$

where,

$$Z' = \begin{bmatrix} S_h(t) \\ I_h(t) \\ R_h(t) \\ S_v(t) \\ I_v(t) \end{bmatrix},$$

$$B = \begin{bmatrix} -\mu_h & 0 & \lambda_h & 0 & 0 \\ 0 & -P_2 & 0 & 0 & 0 \\ u_1+u_2 & \gamma_h & -P_3 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_v - u_3\epsilon_1 & 0 \\ 0 & 0 & 0 & 0 & -P_1+u_3\epsilon_2 \end{bmatrix}, F = \begin{bmatrix} -S_h(\frac{\beta_2 I_v}{1+\alpha_1 I_v} + \frac{\beta_1 I_h(1-u_1(t))}{1+\alpha_2 I_h}) + b_h \\ S_h(\frac{\beta_2 I_v}{1+\alpha_1 I_v} + \frac{\beta_1 I_h(1-u_1(t))}{1+\alpha_2 I_h}) \\ 0 \\ b_v - \frac{\beta_3 S_v I_h}{1+\alpha_2 I_h} \\ \frac{\beta_3 S_v I_h}{1+\alpha_2 I_h} \end{bmatrix},$$

Where, $P_1 = (\delta_v + \gamma_v), P_2 = (\mu_h + \delta_h + \gamma_h), P_3 = (\lambda_h + \mu_h).$

Where, Z' denotes the derivative with respect to time t. The system (8) is a non-linear system with a bounded coefficient. We set,

$$G(Z)=BZ+ F(Z). \quad (9)$$

The second term on the right hand side of (9) satisfies

$$\begin{aligned} |F(Z_1)-F(Z_2) \le H_1(|(S_{1h}(t)-S_{2h}(t))|+H_2|(I_{1h}(t)-I_{2h}(t))|+H_3|(R_{1h}(t)-R_{2h}(t))| \\ +H_4|(S_{1v}(t)-S_{2v}(t))|+H_5|(I_{1v}(t)-I_{2v}(t))|), \\ \le H(|(S_{1h}(t)-S_{2h}(t))|+|(I_{1h}(t)-I_{2h}(t))|+|(R_{1h}(t)-R_{2h}(t))| \\ +|(S_{1v}(t)-S_{2v}(t))|+|(I_{1v}(t)-I_{2v}(t))|), \end{aligned}$$

Where, the positive constant $H = \max(H_1, H_2, H_3, H_4, H_5)$ is independent of the state variables. Also we have

$$|G(Z_1)-G(Z_2)| \le H|Z_1-Z_2|,$$

Where, $H = H_1 + H_2 + H_3 + H_4 + H_5 + \| M \| < \infty$. So, it follows that the function G is uniformly Lipchitz continuous. From the definition of control variables and non-negative initial conditions we can see that a solution of the system (9) exists, [24].

Now, we consider the control system (3) with the initial conditions (2) to show the existence of the control problem. Note that for bounded Lebesgue measurable controls and non-negative initial conditions, non-negative bounded solutions to the state system exists [24]. For the existence of our control problem, we state and prove the following theorem.

Theorem 4.1: There exists an optimal control $u^* = (u_1^*, u_2^*, u_3^*) \in K$ such that $J(u_1^*, u_2^*, u_3^*) = \min_{(u_1,u_2,u_3)\in k} J(u_1, u_2, u_3)$, subject to the control system (3) with the initial conditions (2).

Proof: To prove the existence of an optimal control, we use the result in [25], the control and the state variable are nonnegative values. In this minimizing the problem, the necessary convexity of the objective functional in u_1, u_2, u_3 are satisfied. The set of control variables $(u_1, u_2, u_3) \in K$ is also convex and closed by the definition. The optimal system is bounded which determines the compactness needed for the existence of optimal control. The integrand in the objective functional (4) is $\left(Q_o S_h + Q_1 I_h + Q_2 S_v + Q_3 I_v + \frac{1}{2}(k_o u_1^2 + k_1 u_2^2(t) + k_2 u_3^2(t))\right)$ is convex in the control set K. Also we can easily see that, there exists a constant $\sigma > 1$ and positive numbers ω_1 and ω_2 such that $J(u_1, u_2, u_3) \geq \omega_1(|u_1|^2 + |u_2|^2 + |u_3|^2)^{\frac{\sigma}{2}} - \omega_2$, which is the existence of an optimal control problem. This end the result.

To find the optimal solution, we apply Pontryagin's Maximum Principle [26] given by: If (x,u) is an optimal solution for an optimal control problem, then there exists a nontrivial vector function $\lambda = (\lambda_1, \lambda_2, ...\lambda_n)$ which satisfying the following inequalities.

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial M(t,x,u,\lambda)}{\partial \lambda}, \\ 0 &= \frac{\partial M(t,x,u,\lambda)}{\partial u}, \\ \frac{d\lambda}{dt} &= -\frac{\partial M(t,x,u,\lambda)}{\partial x}. \end{aligned} \tag{10}$$

Now, we apply the necessary conditions to the Hamiltonian M in (7).

Theorem 4.2: Suppose $S_h^*, I_h^*, R_h^*, S_v^*$ and I_v^* be the optimal state solutions with associated optimal control variables (u_1^*, u_2^*, u_3^*) for the optimal control (2)-(3). Then there exists adjoint variables $\lambda_i,\ for\, i = 1,2,....5$ satisfying

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= \mu_h\lambda_1 + (\lambda_2 - \lambda_1)\left(\frac{\beta_2 I_v^*(1-u_1)}{1+\alpha_1 I_v^*} + \frac{\beta_1 I_h^*(1-u_2)}{1+\alpha_2 I_h^*}\right) - \lambda_3(u_1+u_2) - Q_o,\\
\frac{d\lambda_2}{dt} &= (\lambda_1 - \lambda_2)\frac{\beta_1 S_h^*(1-u_2)}{(1+\alpha_2 I_h^*)^2} + (\mu_h + \delta_h + \gamma_h)\lambda_2 - \gamma_h\lambda_3 + (\lambda_4 - \lambda_5)\frac{\beta_3 S_v^*}{(1+\alpha_2 I_h^*)^2} - Q_1,\\
\frac{d\lambda_3}{dt} &= (\lambda_3 - \lambda_1)\lambda_h + \mu_h\lambda_3,\\
\frac{d\lambda_4}{dt} &= \gamma_v\lambda_4 + (\lambda_4 - \lambda_5)\frac{\beta_3 I_h^*}{1+\alpha_2 I_h^*} + u_3\epsilon_1\lambda_4 - Q_2,\\
\frac{d\lambda_5}{dt} &= (\lambda_1 - \lambda_2)\frac{\beta_2 S_h^* I_v^*}{(1+\alpha_1 I_v^*)^2} + \lambda_5(\gamma_v + \delta_v) + u_3\epsilon_2\lambda_5 - Q_3,
\end{aligned} \tag{11}$$

With transversallity conditions (or boundary conditions)

$$\lambda_i(T_{end}) = 0, \qquad i = 1,2,....5. \tag{12}$$

Furthermore, optimal controls $u_1^*, u_2^*\ and\ u_3^*$ are given by

$$u_1^* = \max\left\{\min\left\{\frac{\frac{(\lambda_1 - \lambda_2)\beta_2 S_h^* I_v^*}{(1+\alpha_2 I_v^*)} - \lambda_3 S_h^*}{k_1}, 1\right\}, 0\right\}, \tag{13}$$

$$u_2^* = \max\left\{\min\left\{\frac{\frac{(\lambda_1 - \lambda_2)S_h^*\beta_1 I_h^*}{1+\alpha_2 I_h^*}}{k_2}, 1\right\}, 0\right\}, \tag{14}$$

$$u_3^* = \max\left\{\min\left\{\frac{(\lambda_1\epsilon_1 S_v^* + \epsilon_2\lambda_2 I_v^*\epsilon_2}{k_3}, 1\right\}, 0\right\}. \tag{15}$$

Proof: To find the adjoint equations and the transversality conditions, we use the Hamiltonian (7). By setting

$S_h(t) = S_h^*(t),\ I_h(t) = I_h^*(t),\ R_h(t) = R_h^*(t),\ S_v(t) = S_v^*(t)\ and\ I_v(t) = I_v^*(t)$, and

differentiating the Hamiltonian (7) with respect to $S_h, I_h, R_h, S_v \; and \; I_V$ respectively, we get (11).

Then solving the equations, $\frac{\partial M}{\partial u_1} = 0, \; \frac{\partial M}{\partial u_3} = 0, \; \frac{\partial M}{\partial u_3} = 0$ on the interior of the control set and then using the optimality conditions and also the property of control space K, then we derive (13)-(15).

Here we are calling the formulas (13)-(15) for the characterization of the optimal control. The optimal control and the state are determined by solving the optimality system, which consisting of the state system (1),the adjoint system (11), initial conditions at (2), boundary conditions (12) and the characterization of the optimal control variables (u_1^*, u_2^*, u_3^*) which is given by (12)-(15). In addition, the second derivative of the Lagrangian with respect to u_1^*, u_2^*, u_3^*, respectively are positive, which shows the minimum of the optimal control $u_1^*, u_2^* \; and \; u_3^*$. Substituting the values of $u_1^*, u_2^* \; and \; u_3^*$ in the control system (3), we obtained the following system,

$$\frac{dS_h^*}{dt} = b_h - \mu_h S_h^* - S_h^*\left(\frac{\beta_2 I_v^*\left(1-\max\left\{\min\left\{\frac{\frac{(\lambda_2-\lambda_1)\beta_2 S_h^* I_v^*}{(1+\alpha_1 I_v^*)} - \lambda_3 S_h^*}{k_1}, 1\right\}, 0\right\}\right)}{1+\alpha_1 I_v^*} + \frac{\beta_1 I_h^*\left(1-\max\left\{\min\left\{\frac{\frac{(\lambda_1-\lambda_2)S_h^*\beta_1 I_h^*}{1+\alpha_2 I_h^*}}{k_2}, 1\right\}, 0\right\}\right)}{1+\alpha_2 I_h^*}\right) + \lambda_h R_h^*,$$

$$\frac{dI_h^*}{dt} = S_h^*\left(\frac{\beta_2 I_v^*\left(1-\max\left\{\min\left\{\frac{\frac{(\lambda_2-\lambda_1)\beta_2 S_h^* I_v^*}{(1+\alpha_2 I_v^*)} - \lambda_3 S_h^*}{k_1}, 1\right\}, 0\right\}\right)}{1+\alpha_1 I_v^*} + \frac{\beta_1 I_h^*\left(1-\max\left\{\min\left\{\frac{\frac{(\lambda_1-\lambda_2)S_h^*\beta_1 I_h^*}{1+\alpha_2 I_h^*}}{k_2}, 1\right\}, 0\right\}\right)}{1+\alpha_2 I_h^*}\right) - (\mu_h + \delta_h + \gamma_h) I_h^*,$$

$$\frac{dR_h^*}{dt} = \gamma_h I_h^* - (\mu_h + \lambda_h) R_h^* + S_h^*\left(\max\left\{\min\left\{\frac{\frac{(\lambda_1-\lambda_2)\beta_2 S_h^* I_v^*}{(1+\alpha_2 I_v^*)} - \lambda_3 S_h^*}{k_1}, 1\right\}, 0\right\} + \max\left\{\min\left\{\frac{\frac{(\lambda_1-\lambda_2)S_h^*\beta_1 I_h^*}{1+\alpha_2 I_h^*}}{k_2}, 1\right\}, 0\right\}\right),$$

$$\begin{aligned}
\frac{dS_v^*}{dt} &= b_v - \gamma_v S_v^* - \frac{\beta_3 S_v^* I_h^*}{1+\alpha_2 I_h^*} - \epsilon_1 S_v^*\left(\max\left\{\min\left\{\frac{(\lambda_1\epsilon_1 S_v^* + \epsilon_2\lambda_2 I_v^*\epsilon_2}{k_3}, 1\right\}, 0\right\}\right), \\
\frac{dI_v^*}{dt} &= \frac{\beta_3 S_v^* I_h^*}{1+\alpha_2 I_h^*} - \gamma_v I_v^* - \delta_v I_v^* - \epsilon_2 I_v^*\left(\max\left\{\min\left\{\frac{(\lambda_1\epsilon_1 S_v^* + \epsilon_2\lambda_2 I_v^*\epsilon_2}{k_3}, 1\right\}, 0\right\}\right).
\end{aligned} \tag{16}$$

And

$$\begin{aligned}
M^* &= Q_o S_h^* + Q_1 I_h^* + Q_2 S_v^* + Q_3 I_v^* + \frac{1}{2}\left(k_o \max\left\{\min\left\{\frac{\frac{(\lambda_1-\lambda_2)\beta_2 S_h^* I_v^*}{(1+\alpha_2 I_v^*)} - \lambda_3 S_h^*}{k_o}, 1\right\}, 0\right\}\right)^2 \\
&+ k_1\left(\max\left\{\min\left\{\frac{\frac{(\lambda_1-\lambda_2)S_h^*\beta_1 I_h^*}{1+\alpha_2 I_h^*}}{k_1}, 1\right\}, 0\right\},\right)^2 + k_2\left(\max\left\{\min\left\{\frac{(\lambda_1\epsilon_1 S_v^* + \epsilon_2\lambda_2 I_v^*\epsilon_2}{k_2}, 1\right\}, 0\right\}\right)^2\right) \\
&+ \lambda_1\frac{dS_h^*}{dt} + \lambda_2\frac{dI_h^*}{dt} + \lambda_3\frac{dR_h^*}{dt} + \lambda_4\frac{dS_v^*}{dt} + \lambda_5\frac{dI_v^*}{dt}.
\end{aligned} \tag{17}$$

5. NUMERICAL RESULTS AND DISCUSSION

In this section, we discuss the numerical results of our proposed model (1) and the control system (3). First, we solve the system (1) and then solve the control system (3) with the backward Runge-Kutta order four schemes. Then we solve the adjoint system and the characterized solution is obtained. The Numerical results show that the optimal controls are affective, by using some educational campaign and awareness about the disease spread in the human. The numerical results from Figs. 1 to 8, it is proven that the disease may control. The values of parameters used for the numerical simulation in the optimal control problem are shown in Table 1. The constants used in the objective functional are

$$\epsilon = 0.37, \epsilon_2 = 0.7, k_o = 0.03, k_1 = 0.1, k_2 = 0.003, Q_o = 1, Q_1 = 0.1, Q_2 = 3, Q_3 = 0.9$$

were used. Figs. 1 and 2 shows the population of susceptible and infected individuals in both the system respectively. The bold shows the without control and the dashed line shows without control system. The dashed line in Fig. 1 increase, that the control is effective and we want that, the increase in susceptible population. In Fig. 2 the dashed line shows the control system. The dashed line decreases as compared to bold. The infected individuals in the human population decreases. Fig. 3 and Fig.4 shows the control in both the system with and without control. In Fig. 3 the recovered individuals increases which shows by a dashed line and in Fig. 4 the susceptible vector population decreases which shown by a dashed line in the control system. The bold line shows without control. Fig. 5 shows the control in the infected vector in both the system. The dashed line shows the control system and the bold line shows the control in without control system.

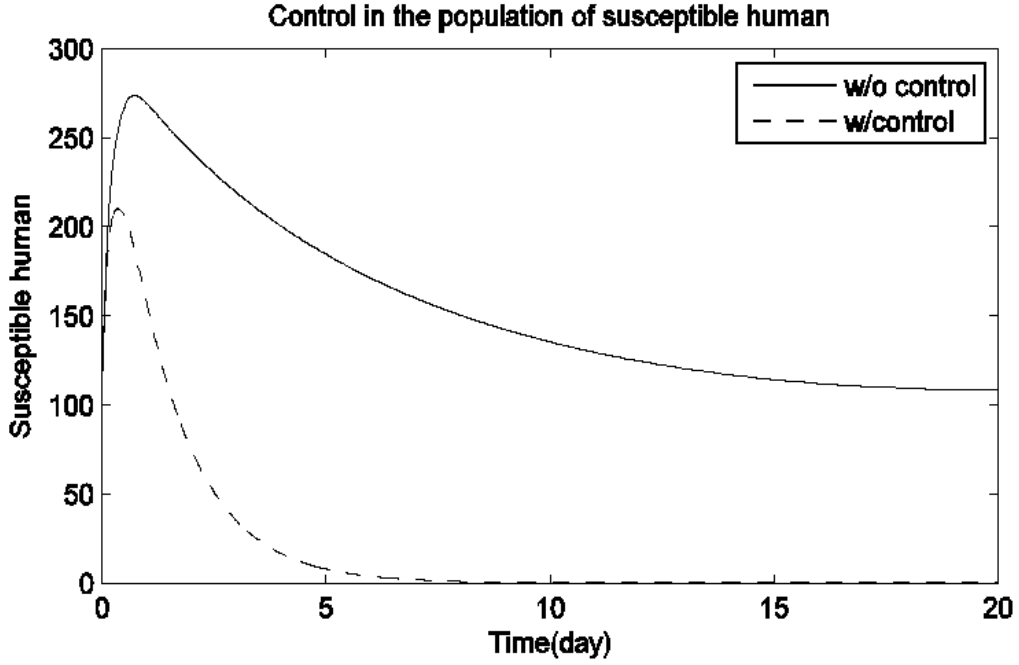

Fig. 1. The plot shows the comparison between the two models with and without control

Table 1. Parameter values used in numerical simulations of the optimal control model

Notation parameter	Value	Source
b_h Recruitment rate for human population	1.2	Assumed
β_1 Transmission rate for human population	0.04	Assumed
β_2 Transmission rate for vector population	0.04	Assumed
β_3 Transmission rate between S_v and I_h	0.04	Assumed
μ_h Natural mortality rate of human population	4.6×10^{-5}	[6]
λ_h Proportionality constant	2.85×10^{-3}	[6]
δ_h Disease death rate for human population	0.4×10^{-3}	[27]
γ_v Natural mortality rate of vector population	1.8×10^{-3}	[6]
		[6]
δ_v Disease death rate for vector population	0.0067	Assumed
α_1 Human inhibitory effect	0.83	Assumed
α_2 Vector inhibitory effect	0.83	Assumed
b_v Recruitment rate for vector population	1.3	Assumed
γ_h Recovery rate from infection	2.7×10^{-3}	[21]

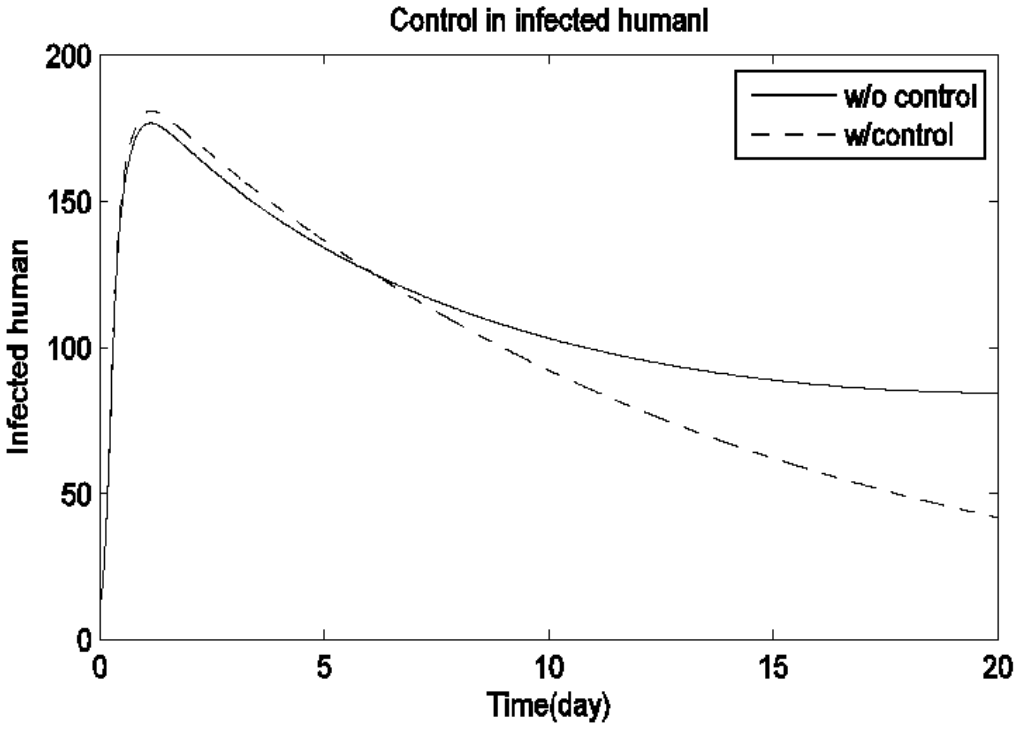

Fig. 2. The plot represents the comparison between the two models with and without control

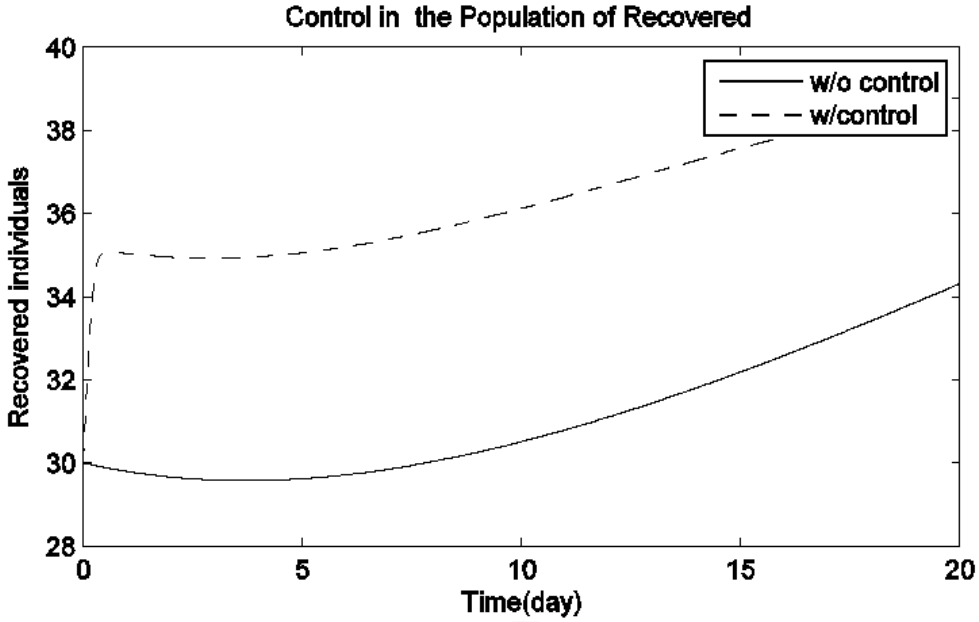

Fig. 3. The plot represents the recovered individual comparison between the two models with and without control

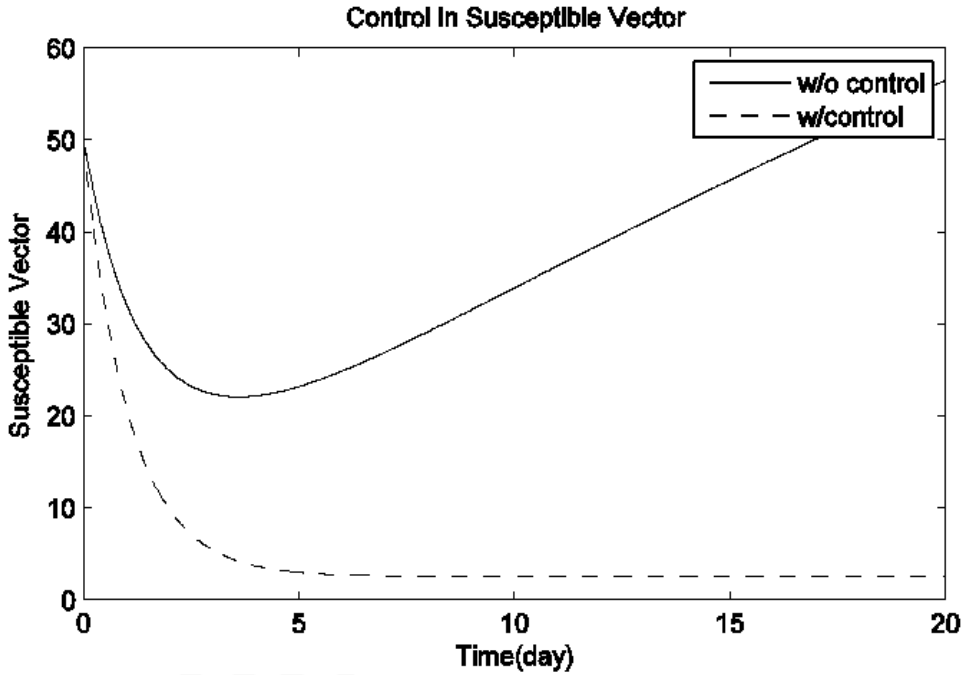

Fig. 4. The plot represents the susceptible vector comparison between the two models with and without control

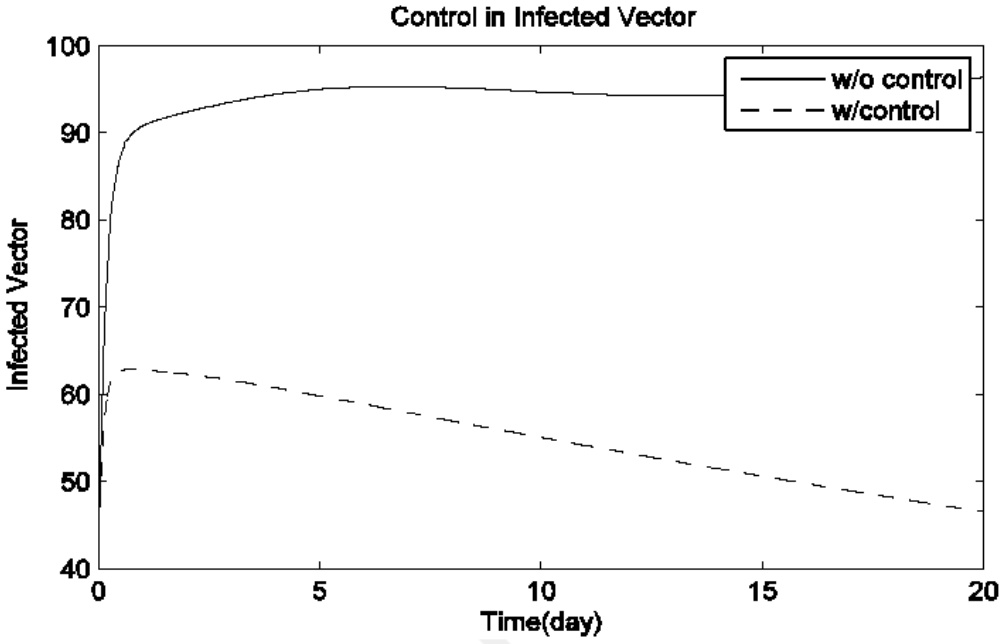

Fig. 5. The plot represents the infected vector comparison between the two models with and without control

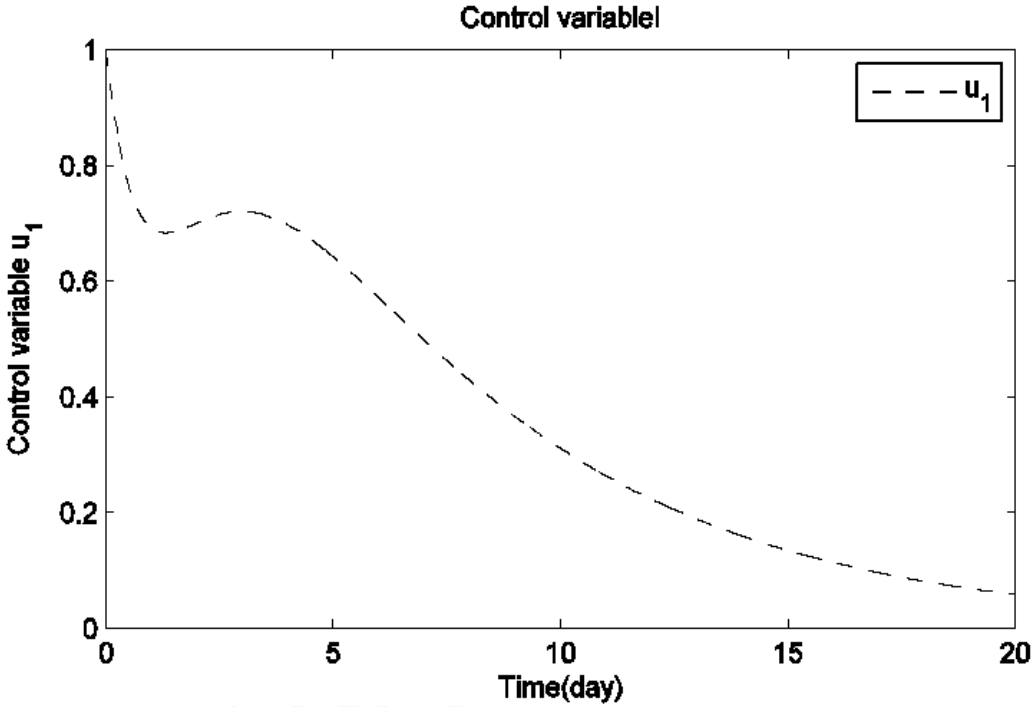

Fig. 6. The plot represents the control variable u1

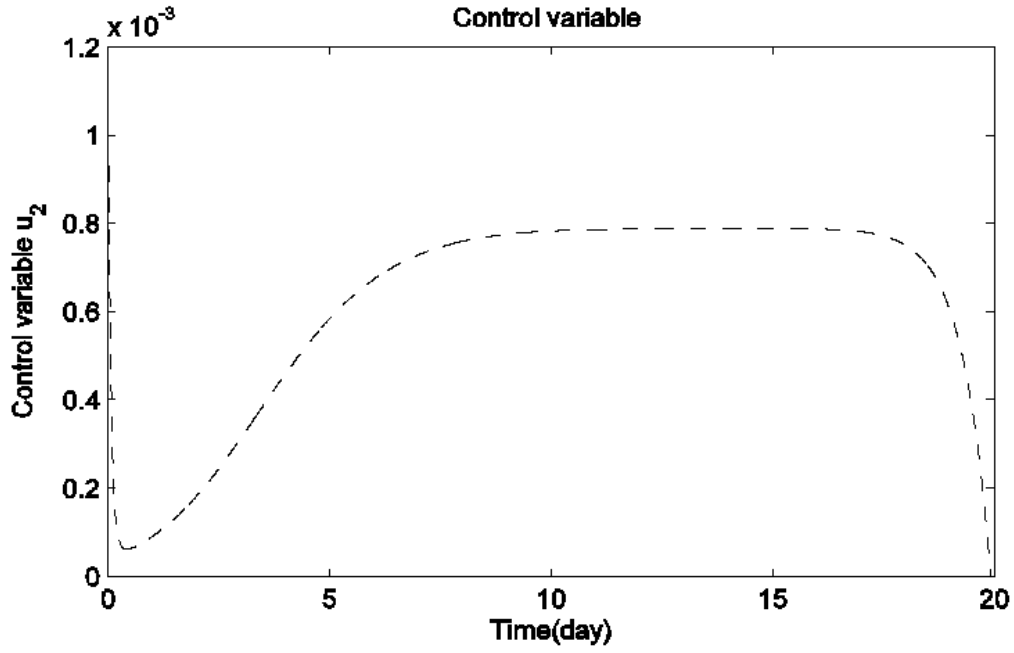

Fig. 7. The plot represents the control variable u2

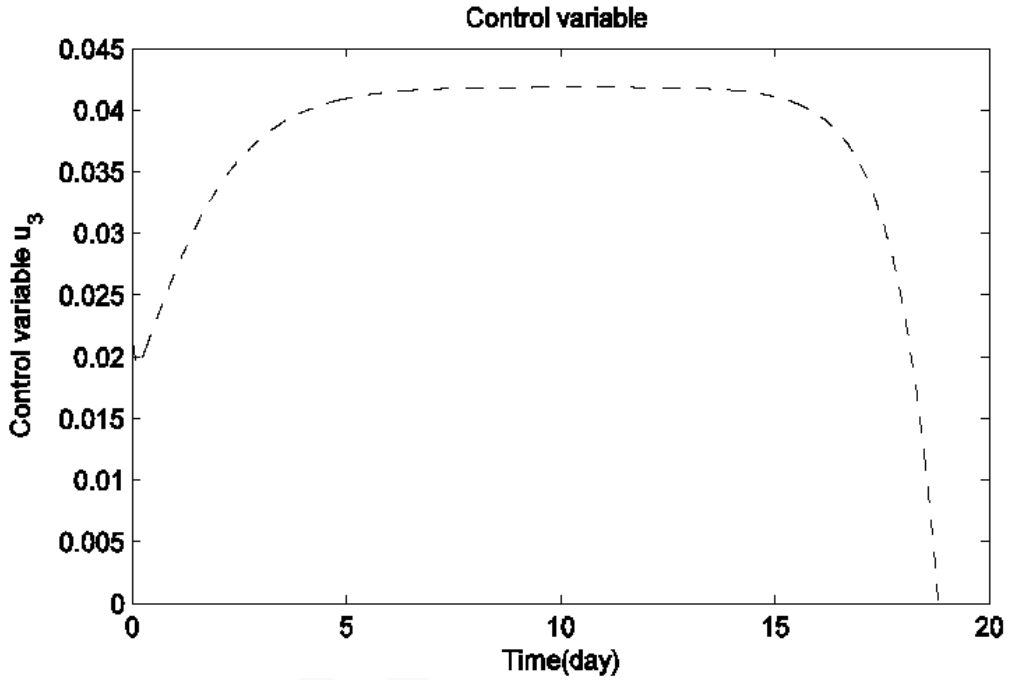

Fig. 8. The plot represents the control variable u3

4. CONCLUSION

We have presented an epidemic model of leptospirosis disease with optimal control techniques. First, we have obtained a mathematical model by applying the optimal control techniques. We have defined the control variables in detail in Section**.** Due to unavailability of the vaccine in the world, except a few countries, like Cuba and China in which human vaccine are available, we have to prevent from the disease by adapting the above mention prevention in the form of control variables $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$.Animal vaccines are only for a few strains. Dog vaccines are effective for at least one year. Currently, no human vaccine is available in the US. Then we define our control set and the Objective functional and then we find the optimality conditions, the control variables characterizations and the adjoint system are obtained. Also the existence of the problem is discussed. In the last, the numerical results of both the system is analyzed for comparison. The optimal control technique is one of the most powerful and efficient techniques that we can completely characterizes the problem. We can see the plot and the variation of the parameters, we can know about the disease eradication in the host population.

ACKNOWLEDGEMENTS

The Authors are thankful to the Editor in Chief and reviewers for their careful reading of the original manuscripts.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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