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# Gerber-Shiu Function in a Discrete-time Risk Model with Dividend Strategy

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors have equal contributions. Both authors have read and approved the final manuscript.*

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## Abstract

In this paper, a discrete-time risk model with dividend strategy and a general premium rate is considered. Under such a strategy, once the insurer's surplus hits a constant dividend barrier $b$, dividends are paid off to shareholders at $\alpha$ instantly. Using the roots of a generalization of Lundberg's fundamental equation and the general theory on difference equations, two difference equations for the Gerber-Shiu discounted penalty function are derived and solved. The analytic results obtained are utilized to derive the probability of ultimate ruin when the claim sizes is a mixture of two geometric distributions. Numerical examples are also given to illustrate the applicability of the results obtained.

*Keywords: Compound binomial model; two-step premium; defective renewal equation; Gerber-Shiu discounted penalty function; dividend strategy.*

## 1 Introduction

Risk theory has a long development time, Lundberg [1] and Gramer [2] established the connection of risk theory. The compound binomial model that was first proposed by Gerber [3] have received considerable attention. For

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instance, Shiu [4], Willmot [5] and Dickson [6] have analyzed the compound binomial model. Markov chain is understood to be a stochastic process in discrete time possessing a certain conditional independence property. The state space may be finite, countably infinite or even more general. Cossette et al. [7] consider the so-called compound Markov binomial model which introduces dependency between claim occurrences. For an generalization of the classical risk model see Landriault [8]. Furthermore, in the discrete time risk model, the issue related to dividend is also widely considered.

Dividend strategies for insurance risk models were first proposed by DeFinetti [9] to reflect more realistically the surplus cash flows in an insurance portfolio. Because of the certainty of ruin for a risk model with a constant dividend barrier, the calculation of the Gerber-Shiu discounted penalty function is a major problem of interest in the context. Among the class of discrete-time risk models, Tan and Yang [10] derived a recursive algorithm to compute a particular class of Gerber-Shiu penalty functions in the framework of the compound binomial model with randomized dividend payments. Landriault [11] then generalized Tan and Yang's model to consider the compound binomial model with a multi-threshold dividend structure and randomized dividend payments. In the discrete time risk model, He and Yang [12] considered that dividends are paid randomly to shareholders and policyholders in the framework of the compound binomial model. In the framework of a discrete semi-Markov risk model, a randomized dividend policy is studied by Yuen et al. [13]. Zhang and Liu [14] consider a discrete-time risk model with a mathematically tractable dependence structure between interclaim times and claim sizes in the presence of an impulsive dividend strategy.

The paper is structured as follows: a brief description of the discrete-time model and the introduction of the Gerber-Shiu discounted penalty function are considered in Section 2. In section 3, we obtain and solve a non-homogeneous difference equation satisfied by the the Gerber-Shiu discounted penalty function $m(u;b)$. Closed-form solutions for $m_b(u)$ are obtained when the claim sizes is a mixture of two geometric distributions and corresponding numerical examples are also provided in Section 4.

# 2 The model

Throughout, denote by $N$ the set of natural numbers and $N^+ = N/\{0\}$. In the compound binomial model, the claim number process $\{N_k, k \in N\}$ is assumed to be a renewal process with independent and identically distributed (i.i.d.) interclaim times $\{W_j, j \in N^+\}$ having probability mass function (p.m.f.) $f_W(j) = q(1-q)^{j-1}$ for $l \in N^+$. Equivalently, the probability of having a claim is $p(0 < p < 1)$ and the probability of no claim is $q = 1 - p$. The individual claim amount r.v.'s(random variables) $\{X_j, j \in N^+\}$ form a sequence of strictly positive, integer-valued and i.i.d. r.v.'s. We suppose that the r.v.'s $\{X_j, j \in N^+\}$ are distributed as a generic r.v. $X$ with p.m.f. $f(x)$, probability generating function (p.g.f.) $\tilde{f}(x)$. Moreover, it is assumed that the r.v.'s $W_1, W_2, \ldots$ and $X_1, X_2, \ldots$ are mutually independent. Let $S_k = \sum_{i=1}^{N_k} X_i$ be the total amount of settled claims up the end of the kth time period with $S_0 = 0$.

Suppose that premiums are received at the beginning of each time period, and claims are paid out at the end of each time period. Denote $u \ge 0$ to be the initial surplus, $b > 0$ the constant barrier level, and $c_1 > 0$ the annual premium . Under such a strategy, let $\alpha(0 < \alpha \le c_1)$ be the annual dividend rate, once the insurer's surplus at time $k$ hits or exceeds a constant dividend barrier $b$, dividends are paid off to shareholders at $\alpha$ instantly. In this case, the net premium after dividend payments is $c_2 = c_1 - \alpha \ge 0$. The corresponding surplus of the insurer at the end of the kth time period is $U_b(k)$ for $k = 1, 2, \cdots$ can be described as

$$U_b(k)=\begin{cases} U_b(k-1)+c_1-\eta_k X_k\ , & U_b(k-1)\le b \\ U_b(k-1)+c_2-\eta_k X_k, & U_b(k-1)>b \end{cases}, \tag{1}$$

where $U_b(0)=u$ , $\{\eta_k, k\in N\}$ is an independent and identically distributed Bernoulli sequence, we denote by $\eta_k=1$ the event of having a claim at the time $k$ and denote by $\eta_k=0$ the event that no claim at the time $k$. We assume that $P(\eta_i=1)=p$ and $P(\eta_i=0)=1-p=q$ and surplus process $U_b(k)$ has a positive drift by letting $c_2>pE[X]$ (known as the positive security loading condition in ruin theory).

Define $\tau_b=\min\{k:U_b(k)<0\}$ to be the time of ultimate ruin. Let $v$ be a constant annual discount rate for each period. When ruin occurs, $U_b(\tau-1)$ is the surplus one period prior to ruin and $|U_b(\tau)|$ is the deficit at ruin. For $v\in(0,1]$, the well-known Gerber-Shiu discounted penalty function is then defined as

$$m(u;b)=E\left\{v^{\tau_b}\omega(U_{\tau_b-1},\left|U_{\tau_b}\right|)I_{\{\tau_b<\infty\}}\middle|U_0=u\right\}, \tag{2}$$

where $\omega: N\times N^+\to R$ is a penalty function and $I_{\{Q\}}$ is the indicator function of an event $Q$. Also, we consider some special cases of (2) with successively simplified the penalty functions. If $\omega(n_1,n_2)=1$ for $(n_1,n_2)\in N\times N^+$, we get the generating function of the time to ruin, i.e.

$$m_b(u)=E\left\{v^{\tau_b}I_{\{\tau_b<\infty\}}\middle|U_0=u\right\}.$$

# 3. The Gerber-Shiu discounted penalty function

In this section, we derive two difference equations for the Gerber-Shiu discounted penalty function: one for the initial surplus below the barrier level $b$ and the other for the initial surplus above the barrier level $b$. Clearly, the Gerber-Shiu discounted penalty function $m(u;b)$ behaves differently, depending on whether its initial surplus $u$ is below or above the barrier level $b$. Hence, we write

$$m(u;b)=\begin{cases} m_1(u), & 0\le u<b \\ m_2(u), & u\ge b \end{cases}.$$

In order to identify the structural form of the solution for the Gerber-Shiu discounted penalty function, three cases will be considered separately.

## 3.1 For initial surpluses less than the barrier $b$

In the first scenario, the initial surplus below the barrier $b$, for $u=0,1,\ldots,b-c_1-1$, we have

$$\begin{aligned} m_1(u)&=vqm_1(u+c_1)+vp\sum_{j=1}^{u+c_1}m_1(u+c_1-j)f(j)+vp\sum_{j=u+c_1+1}^{\infty}\omega(u+c_1;j-u-c_1)f(j) \\ &=vqm_1(u+c_1)+vp(m_1*f)(u+c_1)+\gamma_1(u), \end{aligned} \tag{3}$$

where

$$\gamma_1(u) = vp\sum_{j=u+c_1+1}^{\infty}\omega(u+c_1; j-u-c_1)f(j).$$

and $m_1 * f$ holds for the convolution product of $m_1$ and $f$ .

To state that (3) is a non-homogeneous difference equation of order $c_1$, we re-express (3) according to the forward difference operator $\Delta$ and its property (see Chapter 2 of Kelly & Peterson [15],

$$m(u+c) = \sum_{j=0}^{c}\binom{c}{j}\Delta^j m(u), \tag{4}$$

substituting (4) into (3) shows

$$m_1(u) = vq\sum_{j=0}^{c_1}\binom{c_1}{j}\Delta^j m_1(u) + vp\sum_{j=0}^{c_1}\binom{c_1}{j}\Delta^j (m_1 * f)(u) + \gamma_1(u), \tag{5}$$

for $u = 0,1,\dots,b-c_1-1$, (5) can be simplified to

$$\sum_{j=0}^{c_1}a_{1,j}\Delta^j m_1(u) = \sum_{j=0}^{c_1}b_{1,j}\Delta^j(m_1 * f)(u) + \gamma_1(u), \tag{6}$$

where

$$a_{1,j} = I_{\{j=0\}} - vq\binom{c_1}{j},\ b_{1,j} = vp\binom{c_1}{j}.$$

and $A_1(z)$, $B_1(z)$ are polynomials (in $z$ ) defined as

$$A_1(z) = \sum_{j=0}^{c_1}a_{1,j}z^j, B_1(z) = \sum_{j=0}^{c_1}b_{1,j}z^j .$$

becomes

$$A_1(\Delta)m_1(u) = B_1(\Delta)(m_1 * f)(u) + \gamma_1(u), \qquad u = 0,1,2,\dots,b-c_1-1, \tag{7}$$

We know from (7) that $m_1(u)$ satisfies a non-homogeneous difference equation of order $c_1$. From the general theory on difference equations, every solution to a $c_1$-th order difference equation can be expressed as a particular solution to this difference equation plus a linear combination of $c_1$ linearly independent solutions to the associated homogeneous difference equation (cf. Elaydi [16], Theorem 2.30). Therefore, for $u = 0,1,\dots,b-1$, the Gerber-Shiu discounted penalty function can be expressed as

$$m_1(u) = \phi_1(u) + \sum_{j=0}^{c_1-1}\alpha_{1,j}y_{1,j}(u), \quad u = 0,1,2,\dots,b-1. \tag{8}$$

where $\{y_{1,j}(u)\}_{u=0}^{\infty}$ $(j=0,1,...,c_1-1)$ are $c_1$ fundamental solutions to the following homogeneous difference equation

$$A_1(\Delta)y_1(u)=B_1(\Delta)(y_1*f)(u) \qquad u\geq 0. \tag{9}$$

$\{\phi_1(u)\}_{u=0}^{\infty}$ is a particular solution to

$$A_1(\Delta)\phi_1(u)=B_1(\Delta)(\phi_1*f)(u)+\gamma_1(u) \qquad u\geq 0. \tag{10}$$

Combining (3) and (9), we get

$$y_1(u)=vqy_1(u+c_1)+vp(y_1*f)(u+c_1). \tag{11}$$

Multiplying (11) by $z^{u+c_1}$ and then summing over $u$ from $0$ to $\infty$ lead to

$$\sum_{u=0}^{\infty}z^{u+c_1}y_1(u)=vq\sum_{u=0}^{\infty}z^{u+c_1}y_1(u+c_1)+vp\sum_{u=0}^{\infty}z^{u+c_1}(y_1*f)(u+c_1), \tag{12}$$

routine calculations lead to

$$z^{c_1}\tilde{y}_1(z)=vq\left[\tilde{y}_1(z)-\sum_{u=0}^{c_1-1}z^u y_1(u)\right]+vp\left\{\tilde{y}_1(z)\tilde{f}(z)-\sum_{u=0}^{c_1-1}z^u(y_1*f)(u)\right\}.$$

After some algebra, one could see that (12) can be written as

$$\tilde{y}_1(z)=\frac{-v\left\{q\sum_{u=0}^{c_1-1}z^u y_1(u)+p\sum_{u=0}^{c_1-1}z^u(y_1*f)(u)\right\}}{z^{c_1}-vq-vp\tilde{f}(z)}. \tag{13}$$

By choosing $y_{1,j}(u)=I_{\{j=u\}}$ for $j,u\in\{0,1,...,c_1-1\}$. According to (13), the generating function associated to the fundamental solution $\{y_{1,j}(u)\}_{u=o}^{\infty}$ is

$$\tilde{y}_{1,j}(z)=\frac{-v\left\{qz^j+p\sum_{u=j+1}^{c_1-1}z^u f(u-j)\right\}}{z^{c_1}-vq-vp\tilde{f}(z)}=\frac{-R_{1,j}(z)}{\tilde{h}_{1,1}(z)-\tilde{h}_{1,2}(z)} \qquad u\geq 0, \tag{14}$$

where

$$\tilde{h}_{1,1}(z)=z^{c_1},\ \tilde{h}_{1,2}(z)=vq+vp\tilde{f}(z), R_{1,j}(z)=v\left\{qz^j+p\sum_{u=j+1}^{c_1-1}z^u f(u-j)\right\}.$$

**Lemma 3.1** : When $v\in(0,1)$, the denominator in (14) has exactly $c_1$ zeros, say $\{z_i\}_{i=1}^{c_1}$ inside the unit circle $C=\{z:|z|=1\}$.

**Lemma 3.2** : When $v=1$, the denominator in (14) has exactly $c_1-1$ zeros, say $\{z_i\}_{i=1}^{c_1-1}$ inside the unit circle $C=\{z:|z|=1\}$ and another trivial root $z_{c_1}=1$ .

For the rest of the paper, we assume that all $\{z_i\}_{i=1}^{c_1}$ are distinct, since the analysis of the multiple roots of Lundberg's generalized equation leads to tedious derivations.

Let $\pi_i(z)=\prod_{j=1}^{c_i}(z-z_j)$ and $\pi_i^{'}(z_k)=\prod_{l=1,l\neq k}^{c_i}(z_k-z_l)$, from Liu and Bao [17], we have

$$\frac{\widetilde{h}_{1,1}(z)-\widetilde{h}_{1,2}(z)}{\pi_1(z)}=1-vpT_zT_{z_{c_1}}\ldots T_{z_2}T_{z_1}f(c_1), \tag{15}$$

where $T_z$ is an operator (see Li [18] )defined as

$$T_z y(c)=\sum_{u=0}^{\infty}z^u y(u+c)=\sum_{u=c}^{\infty}z^{u-c}y(u).$$

(14) can be rewrote as

$$\widetilde{y}_{1,j}(z)=\frac{\dfrac{-R_{1,j}(z)}{\pi_1(z)}}{\dfrac{\widetilde{h}_{1,1}(z)-\widetilde{h}_{1,2}(z)}{\pi_1(z)}}. \tag{16}$$

Regarding the numerator in (16), partial fractions yield the equivalent representation

$$\frac{-R_{1,j}(z)}{\pi_1(z)}=\sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)}\cdot\frac{1}{z_k-z}. \tag{17}$$

By inserting (15) and (17) into (16), we obtain

$$\widetilde{y}_{1,j}(z)=v\,p\widetilde{y}_{1,j}(z)T_zT_{z_{c_1}}\ldots T_{z_2}T_{z_1}f(c_1)+\sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi^{'}(z_k)}\frac{1}{z_k-z}. \tag{18}$$

**Theorem 3.1** : For $j=0,1,\cdots,c_1-1$, $y_{1,\ j}(u)$ satisfies the following defective renewal equation

$$y_{1,j}(u)=\varsigma_1\sum_{n=0}^{u}y_{1,j}(u-n)\chi_1(n)+\zeta_1(u), \tag{19}$$

where

$$\varsigma_1=vpT_1T_{z_{c_1}}\ldots T_{z_2}T_{z_1}f(c_1),\ \chi_1(n)=\frac{T_{z_{c_1}}\ldots T_{z_2}T_{z_1}f(c_1+n)}{T_1T_{z_{c_1}}\ldots T_{z_2}T_{z_1}f(c_1)},\ \zeta_1(u)=\sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)}\left(\frac{1}{z_k}\right)^{u+1}.$$

Now, we turn our attention to the calculation of the particular solutions $\{\phi_1(u)\}_{u=0}^{\infty}$, combining (3) and (10), $\{\phi_1(u)\}_{u=0}^{\infty}$ satisfies

$$\phi_1(u) = vq\phi_1(u+c_1) + vp(\phi_1 * f)(u+c_1) + \gamma_1(u). \tag{20}$$

We use a solution procedure analogous to the fundamental solutions, we get

$$\begin{aligned}\tilde{\phi}_1(z) &= \frac{z^{c_1}\tilde{\gamma}_1(z) - v\left\{q\sum_{u=0}^{c_1-1} z^u \phi_1(u) + p\sum_{u=0}^{c_1-1} z^u (\phi_1 * f)(u)\right\}}{z^{c_1} - vq - vp\tilde{f}(z)} \\ &= \frac{z^{c_1} T_z \gamma_1(0) - Q_{1,j}(z)}{z^{c_1} - vq - vp\tilde{f}(z)}.\end{aligned} \tag{21}$$

where $Q_{1,j}(z) = v\left\{q\sum_{u=0}^{c_1-1} z^u \phi_1(u) + p\sum_{u=0}^{c_1-1} z^u (\phi_1 * f)(u)\right\}$ is a polynomial of degree $c_1 - 1$ (or less) in $z$. It is known from (35) in Liu and Zhang [14] that

$$\frac{z^{c_1} T_z \gamma_1(0) - Q_{1,j}(z)}{\pi_1(z)} = T_z T_{z_{c_1}} \ldots T_{z_2} T_{z_1} \gamma_1(0). \tag{22}$$

By substituting (15) and (22) into (21), we get

$$\tilde{\phi}_1(z) = \frac{T_z T_{z_{c_1}} \ldots T_{z_2} T_{z_1} \gamma_1(0)}{1 - vpT_z T_{z_{c_1}} \ldots T_{z_2} T_{z_1} f(c_1)} = \frac{\tilde{\vartheta}_1(z)}{1 - vpT_z T_{z_{c_1}} \ldots T_{z_2} T_{z_1} f(c_1)}. \tag{23}$$

**Theorem 3.2**：For $u \in N$, it holds that

$$\phi_1(u) = \varsigma_1 \sum_{n=0}^{u} \phi_1(u-n)\chi_1(n) + \vartheta_1(u), \tag{24}$$

where $\vartheta_1(u) = T_{z_{c_1}} \ldots T_{z_2} T_{z_1} \gamma_1(u)$.

In the second scenario, for $u = b - c_1, \cdots, b-1$,

$$m_1(u) = vqm_2(u+c_1) + vp(m * f)(u+c_1) + \gamma_1(u). \tag{25}$$

## 3.2 For initial surpluses equal to or more than the barrier $b$

The last scenario, for $u \geq b$,

$$m_2(u) = vqm_2(u+c_2) + vp(m * f)(u+c_2) + \gamma_2(u), \tag{26}$$

where $\gamma_2(u) = vp \sum_{j=u+c_2+1}^{\infty} \omega(u+c_2; j-u-c_2) f(j)$.

The structural form (8) for $m_1(u)$ is expressed in terms of the $\alpha_{1,j}$, and also depends on $m_2(u)$ in (26). In order to drive the solutions of $m(u)$, shifting the argument $u$ in (26) by $b$ units, for $u \geq 0$ (26) can be rewritten as

$$m_2(u+b) = vqm_2(u+b+c_2) + vp \sum_{j=b}^{u+b+c_2} m_2(j) f(u+b+c_2-j)$$
$$+ vp \sum_{j=0}^{b-1} m_1(j) f(u+b+c_2-j) + \gamma_2(u+b),$$

Let $\xi_2(u) \equiv m_2(u+b)$, (26) becomes

$$\xi_2(u) = vq\xi_2(u+c_2) + vp(\xi_2 * f)(u+c_2) + \eta(u), \quad (27)$$

where $\eta(u) = vp \sum_{j=0}^{b-1} m_1(j) f(u+b+c_2-j) + \gamma_2(u+b)$.

We use a solution procedure analogous to that of Section 3.1, $\xi_2(u)$ satisfies

$$A_2(\Delta)\xi_2(u) = B_2(\Delta)(\xi_2 * f)(u) + \eta(u) \qquad u \geq 0, \quad (28)$$

where

$$A_2(z) = \sum_{j=0}^{c_2} a_{2,j} z^j, B_2(z) = \sum_{j=0}^{c_2} b_{2,j} z^j, a_{2,j} = I_{\{j=0\}} - vq\binom{c_2}{j}, b_{2,j} = vp\binom{c_2}{j}.$$

From the general theory on difference equations, can be expressed as

$$m_2(u+b) \equiv \xi_2(u) = \phi_2(u) \quad u = 0,1,\ldots$$

where $\{\phi_2(u)\}_{u=0}^{\infty}$ satisfies

$$A_2(\Delta)\phi_2(u) = B_2(\Delta)(\phi_2 * f)(u) + \eta(u) \qquad u \geq 0. \quad (29)$$

Some solution procedures are omitted, similar discussions can be find in Section 3.1.Generating function of the particular solution $\phi_2(u)$ is

$$\tilde{\phi}_2(z) = \frac{z^{c_2}\tilde{\eta}(z) - v\left\{q\sum_{u=0}^{c_2-1} z^u \phi_2(u) + p\sum_{u=0}^{c_2-1} z^u (\phi_2 * f)(u)\right\}}{z^{c_2} - vq - vp\tilde{f}(z)} \quad (30)$$
$$= \frac{z^{c_2} T_z \eta(0) - R_{2,j}(z)}{\tilde{h}_{2,1}(z) - \tilde{h}_{2,2}(z)},$$

Where

$$\tilde{h}_{2,1}(z) = z^{c_2}, \ \tilde{h}_{2,2}(z) = vq + vp\tilde{f}(z), R_{2,j}(z) = v\left\{ q\sum_{u=0}^{c_2-1} z^u \phi_2(u) + p\sum_{u=0}^{c_2-1} z^u (\phi_2 * f)(u) \right\}.$$

**Theorem 3.3**: For $u \in N$, it holds that

$$\phi_2(u) = \varsigma_2 \sum_{n=0}^{u} \phi_2(u-n)\chi_2(n) + \vartheta_2(u), \tag{31}$$

where

$$\varsigma_2 = vpT_1 T_{z_{c_2}} \dots T_{z_2} T_{z_1} f(c_2), \ \ \chi_2(n) = \frac{T_{z_{c_2}} \dots T_{z_2} T_{z_1} f(c_2+n)}{T_1 T_{z_{c_2}} \dots T_{z_2} T_{z_1} f(c_2)}, \ \ \vartheta_2(u) = T_{z_{c_2}} \dots T_{z_2} T_{z_1} \eta(u).$$

So for $u \ge b$,

$$m_2(u) \equiv \xi_2(u-b) = \phi_2(u-b). \tag{32}$$

# 4 Numerical results

It is well-known that $f(x) = (1-\rho)\rho^{x-1}$ is a geometric distribution. In this section, it is further assumed that $f(x)$ is a mixture of two geometric distributions with $f(x) = \theta(1-\rho_1)\rho_1^{x-1} + (1-\theta)(1-\rho_2)\rho_2^{x-1}$.

Obviously, probability generating function is $\tilde{f}(x) = \dfrac{z[(1-\rho_1)(1-\rho_2) + \beta(1-z)]}{(1-\rho_1 z)(1-\rho_2 z)}$ where $\beta = \theta\rho_2(1-\rho_1) + (1-\theta)\rho_1(1-\rho_2)$, and mean is $\mu = \dfrac{\theta}{1-\rho_1} + \dfrac{1-\theta}{1-\rho_2}$, we rewrite (16) as

$$\tilde{y}_{1,j}(z) = \frac{-R_{1,j}(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_1(z)}, \tag{33}$$

where

$$\Lambda_i(z) = z^{c_i}(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp[z(1-\rho_1)(1-\rho_2) + \beta(1-z)] \ (i=1,2)$$

Since $\Lambda_i(z)$ is a polynomial of degree $c_i + 1$, with leading coefficient $\rho_1\rho_2$, it can be expressed as

$$\Lambda_i(z) = \rho_1\rho_2\pi_i(z)\prod_{j=1}^{\hbar}(z-\xi_j),$$

where $\xi_i$ are solutions of $\Lambda_i(z)$ on the complex plane. It is notable that $\xi_i$ have a module larger than 1,from performing partial fraction, we have

$$\frac{\pi_i(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_i(z)} = \frac{\pi_i(z)(1-\rho_1 z)(1-\rho_2 z)}{\rho_1\rho_2\pi_i(z)\prod_{j=1}^{\hbar}(z-\xi_j)} = 1+\sum_{i=1}^{\hbar}\frac{\omega_i}{\xi_i - z}, \qquad (34)$$

where

$$\omega_i = \frac{\prod_{k=1}^{\hbar}\left(\rho_k^{-1}-\xi_i\right)}{\prod_{k=1,k\neq i}^{\hbar}\left(\xi_k-\xi_i\right)}.$$

For $i=1$, substituting (34) into (16) shows

$$\tilde{y}_{1,j}(z) = \frac{(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_1(z)}\sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)}\frac{\pi_1(z)}{z_k - z} = \sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)}\left(1+\sum_{i=1}^{\hbar}\frac{\omega_i}{\xi_i - z}\right)\frac{1}{z_k - z}. \quad (35)$$

Upon inversion, we obtain from (35) that

$$\begin{aligned} y_{1,j}(u) &= \sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)}\left[z_k^{-(u+1)} + \sum_{i=1}^{\hbar}\omega_i\sum_{l=0}^{u}\xi_i^{-(u+1-l)}z_k^{-(l+1)}\right] \\ &= \sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)}\left(1-\sum_{i=1}^{\hbar}\frac{\omega_i}{z_k-\xi_i}\right)z_k^{-(u+1)} + \sum_{k=1}^{c_1}\frac{R_{1.j}(z_k)}{\pi_1^{'}(z_k)}\sum_{i=1}^{\hbar}\frac{\omega_i}{z_k-\xi_i}\xi_i^{-(u+1)}, \end{aligned} \qquad (36)$$

Use the same method, we obtain from (30) and (34) that,

$$\tilde{\phi}_i(z) = \frac{\pi_i(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_i(z)}\tilde{\vartheta}_i(z) = \left(1+\sum_{j=1}^{\hbar}\frac{\omega_j}{\xi_j - z}\right)\tilde{\vartheta}_i(z), \qquad (37)$$

Upon the inversion of the generating functions, one obtains from (37) that

$$\phi_i(u) = \vartheta_i(u) + \sum_{j=1}^{\hbar}\omega_j\sum_{l=0}^{u}\xi_j^{-(u+1-l)}\vartheta_i(l). \qquad (38)$$

**Example** : Suppose $c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, v = 0.95, \rho_1 = 0.3, \rho_2 = 0.6$, from (33)

$$\Lambda_1(z) = z^2(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp\left[z(1-\rho_1)(1-\rho_2) + \beta(1-z)\right],$$

$$\Lambda_2(z) = z(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp\left[z(1-\rho_1)(1-\rho_2) + \beta(1-z)\right].$$

And the relatively safety loading condition $c_2 - p\mu > 0$ holds for all $\theta \in (0,1)$. Hence, $\theta$ is chosen to be $0.1, 0.3, 0.5, 0.7, 0.9$, respectively. By solving Lundberg's equation $\Lambda_1(z) = 0$, we obtain the values of $z_i$ 's and $\xi_j$ 's, see Table 1. By solving Lundberg's equation $\Lambda_2(z) = 0$, we obtain the values of $z_1$ 's and $\xi_j$ 's, see Table 2.

**Table 1. Numerical results of $z_i$ 's and $\xi_j$ 's, for $c_1 = 2$.**

| $\theta$ | $z_1$ | $z_2$ | $\xi_1$ | $\xi_2$ |
|---|---|---|---|---|
| 0.1 | -0.8738391540 | 0.9674521058 | 1.5385860914 | 3.3678009567 |
| 0.3 | -0.8801119231 | 0.9682497475 | 1.5521641647 | 3.5969801084 |
| 0.5 | -0.8863334780 | 0.9690110686 | 1.5658315556 | 3.3514908538 |
| 0.7 | -0.8925049154 | 0.9697383777 | 1.5795907333 | 3.3431758044 |
| 0.9 | -0.8986272938 | 0.9704338015 | 1.5934445298 | 3.3347489625 |

**Table 2. Numerical results of $z_1$ 's and $\xi_2$ 's, for $c_2 = 1$.**

| $\theta$ | $z_1$ | $\xi_1$ | $\xi_2$ |
|---|---|---|---|
| 0.1 | 0.916804131469313 | 1.37945341276706 | 3.46374245576363 |
| 0.3 | 0.921092162877287 | 1.40497341947108 | 3.43393441765163 |
| 0.5 | 0.925035115678722 | 1.43171692276687 | 3.4032479615544 |
| 0.7 | 0.928664417778473 | 1.45973526774272 | 3.37160031447881 |
| 0.9 | 0.932009119109698 | 1.48909606873408 | 3.33889481215622 |

Explicit expressions for $y_{1,j}(u)$ is determined by (36), so we obtain the values of $y_{1,j}(u)$ for $\theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10$.

For instance, one has for $\theta = 0.5$,

$$y_0(u) = -0.42464 \times (-0.88633)^{-u} + 0.47519 \times 0.96901^{-u} - 0.05183 \times 1.56583^{-u} + 0.00129 \times 3.40325^{-u}$$

$$y_1(u) = 0.42071 \times (-0.88633)^{-u} + 0.55572 \times 0.96901^{-u} - 0.04023 \times 1.56583^{-u} + 0.00056 \times 3.40325^{-u}$$

Then solve a system of linear equations with $\alpha_{1,j}$, Table 3 lists the values of $\alpha_{1,j}$ 's.

**Table 3. Numerical results of $\alpha_{1,j}$ for $b = 10$.**

| $\theta$ | **0.1** | **0.3** | **0.5** | **0.7** | **0.9** |
|---|---|---|---|---|---|
| $\alpha_{1,0}$ | 0.001602 | 0.00103 | 0.00059 | 0.00027 | $5.109949 \times 10^{-5}$ |
| $\alpha_{1,1}$ | 0.001703 | 0.00109 | 0.00063 | 0.00029 | $6.318151 \times 10^{-5}$ |

Explicit expressions for $\phi_2(u)$ is determined by (38) so we get the values of $\phi_2(u)$ for $\theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10$, see Table 4.

**Table 4. Numerical results of $\phi_2(u)$ for $b = 10, \theta = 0.5$.**

| $u$ | **10** | **11** | **12** | **13** | **14** |
|---|---|---|---|---|---|
| $\phi_2(u)$ | $7.22942 \times 10^{-4}$ | $5.02029 \times 10^{-4}$ | $3.49790 \times 10^{-4}$ | $2.44063 \times 10^{-4}$ | $1.70395 \times 10^{-4}$ |
| $u$ | **15** | **16** | **17** | **18** | **19** |
| $\phi_2(u)$ | $1.8993 \times 10^{-4}$ | $8.31054 \times 10^{-5}$ | $5.80441 \times 10^{-5}$ | $4.05411 \times 10^{-5}$ | $2.83162 \times 10^{-5}$ |

Especially, when $\omega(N_1, N_2) = 1, b = 10$, Fig. 1 and Fig. 2 depict the generating function of the time to ruin $m_b(u)$ as functions of $u$. Observing Fig. 1 and Fig. 2, for each fixed $\theta$ it is easy that a larger $u$ corresponds to a smaller expected ruin time and $m_b(u)$ is a decreasing function of $\theta$ when $u$ is fixed.

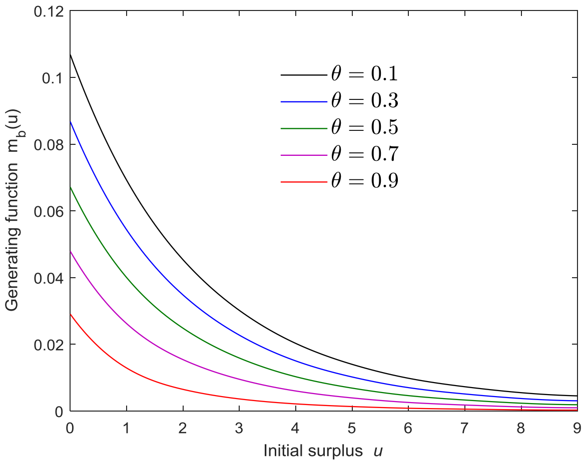


**Fig. 1. Numerical results of $m_b(u)$ for $b = 10, u < b$**

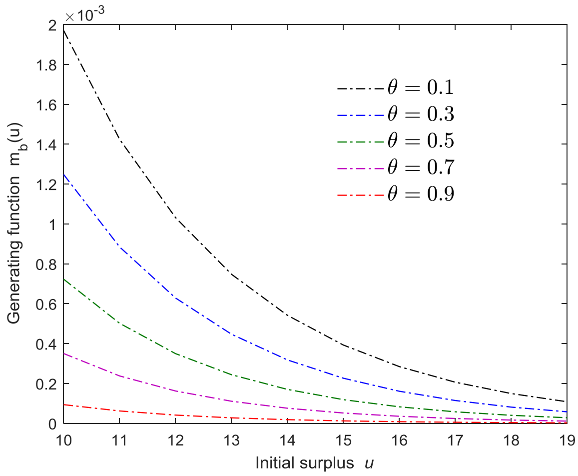


**Fig. 2. Numerical results of $m_b(u)$ for $b = 10, u \geq b$**

## 5. Conclusion

In this paper, we consider the compound binomial model with general premium rate and a constant dividend barrier.Using the roots of a generalization of Lundberg's fundamental equation and the general theory on difference equations, we derive an explicit expression for the Gerber-Shiu discounted penalty function up to the time of ruin. In particular, a numerical example is provided to show that the formulae are readily programmable in practice. From the numerical example given above, we can see that the barrier level has a negative effect on the total Gerber-Shiu discounted penalty function.


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## Competing Interests

Authors have declared that no competing interests exist.

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