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The Copoun Distribution and Its Mathematical Properties

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Due to the ever growing demand for the development of new lifetime distributions to meet the goodness of fit demand of complex datasets, two-parameter distributions has been proposed in recent times. This study therefore aims to contribute to this demand. We propose a new two-parameter lifetime distribution known as the Copoun distribution. Important mathematical properties of the new distribution such as the moments and other related measures, and moment generating function were derived. Finally, the values of the mean, standard deviation, coefficient of variation, skewness, and kurtosis of the Copoun distribution shows that the distribution has the tendency to shift to higher values overall (increasing mean) and narrow around this increased central tendency (decreasing spread, variation and increasing peakedness).

Keywords: *Copoun distribution; lifetime distributions; two-parameter distributions; quantile function; risk measurement function; reliability indices.*

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1 Introduction

Lifetime distributions, also known as survival distributions or failure time distributions provide a statistical framework for modeling the time until an event of interest occurs [1,2]. They provide a powerful tool for analyzing and interpreting data, enabling researchers and practitioners to make informed decisions based on probabilistic models. Due to the complexities of real life dataset, the need for developing new distributions cannot be overemphasized. This is because there is no one size fit all lifetime distribution for any given complex real life dataset. The quest to get a perfect fit to a real life complex dataset has led to the development of several one-parameter lifetime distributions to model real lifetime situations across several fields including engineering, finance, biology, epidemiology, and social sciences.

Examples of the one-parameter lifetime distributions that has been proposed over the years includes the Exponential distribution [3], Lindley distribution [4], Akash distribution [5], Sujatha distribution [6], Ishita distribution [7], Akshaya distribution [8]. Others include Rama distribution [9], Pranav distribution [10], Odoma distribution [11], Nwikpe distribution [12], Iwueze distribution [13], Juchez distribution [14], Chris-Jerry distribution [15].

In recent years, new two-parameter distributions have emerged in the literature. These new two-parameter distributions have been shown to provide better fit to complex real life datasets than the one-parameter distributions. Some of the recently developed two-parameter distributions includes the Darna distribution [16], the Hamza distribution [17], the Samade distribution [18], and the Alzoubi distribution [19].

It is important to note that these distributions are a mixture of the Exponential and Gamma distributions. These two distributions are known to have their weaknesses. The weakness of the Exponential distribution is that the hazard rate function is constant; hence, it cannot handle datasets with monotone non-decreasing hazard rates [13,3,20,21,22]. Furthermore, the weakness of the Gamma distribution is that the survival rate function cannot be expressed in closed form [13,5,23]. The limitations of these two distributions are what the aforementioned one-parameter and two-parameter distributions address, providing distributions whose survival rate function can be expressed in closed form and hazard rate functions capable of handling datasets with monotone non-decreasing hazard rates.

In this study, we propose a new two-parameter distribution called the Copoun distribution. The subsequent sections of the paper will be arranged as follows. Section 2 discusses the new distribution, section 3 discusses the mathematical properties of the Copoun distribution, and section 4 concludes the paper.

2 The Copoun Distribution

This section will introduce the probability density function (pdf) and the cumulative distribution function (cdf) of the Copoun distribution and illustrate the different shapes of the Copoun distribution.

Definition: A random variable X is said to have a Copoun distribution (CpD) with parameters η and ϕ if its probability density function, is given by

$$g(x;\eta,\phi) = \frac{\eta^2}{(\phi+\eta)}\left[1 + \frac{\phi\eta^2 x^3}{6}\right]e^{-\eta x};\ x > 0, \eta > 0, \phi > 0 \tag{2.1}$$

Remark 1: The pdf in equation 1 is a two component density of an Exponential (η) and Gamma $(4, \eta)$ distribution with mixing proportions $\pi_1 = \frac{\eta}{(\phi+\eta)}$ and $\pi_2 = \frac{\phi}{(\phi+\eta)}$ such that

$$g(x;\eta,\phi) = \pi_1 g_1(x;\eta) + \pi_2 g_2(x;\eta) \tag{2.2}$$

The corresponding cdf of equation 1 is given by

$$G(x;\eta,\phi) = 1 - \left[1 + \frac{\phi\eta^3 x^3 + \phi\eta^2 x^3 + \phi\eta x}{6(\phi+\eta)}\right]e^{-\eta x} \tag{2.3}$$

Remark 2: It can be easily seen that the pdf in equation 1 is a proper pdf.

The graphical plots of the theoretical density and distribution function (for some selected but different real points of η and ϕ) of a Coupon distribution are shown in the Fig. 1 and Fig. 2 below.

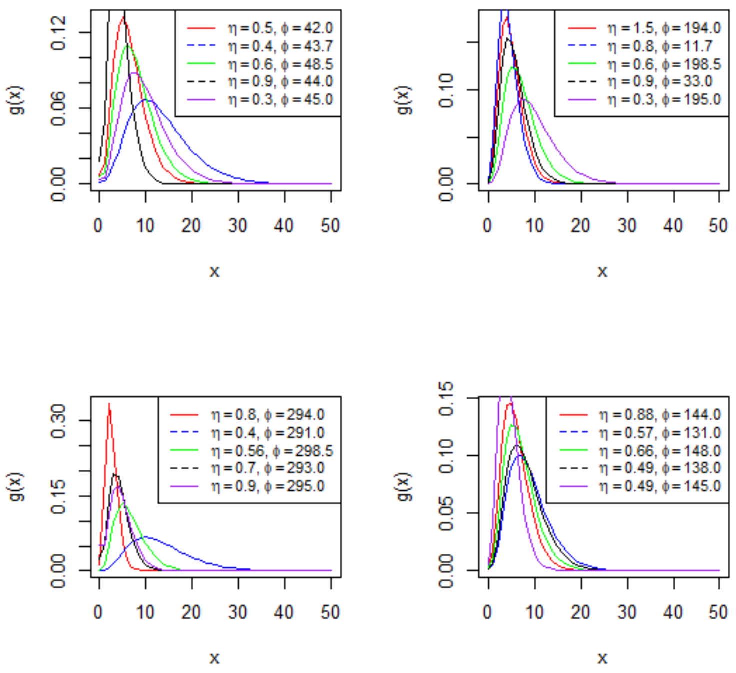

Fig. 1. The graphical plots of the probability density function (for some selected but different real points of η and ϕ) of a Coupon distribution

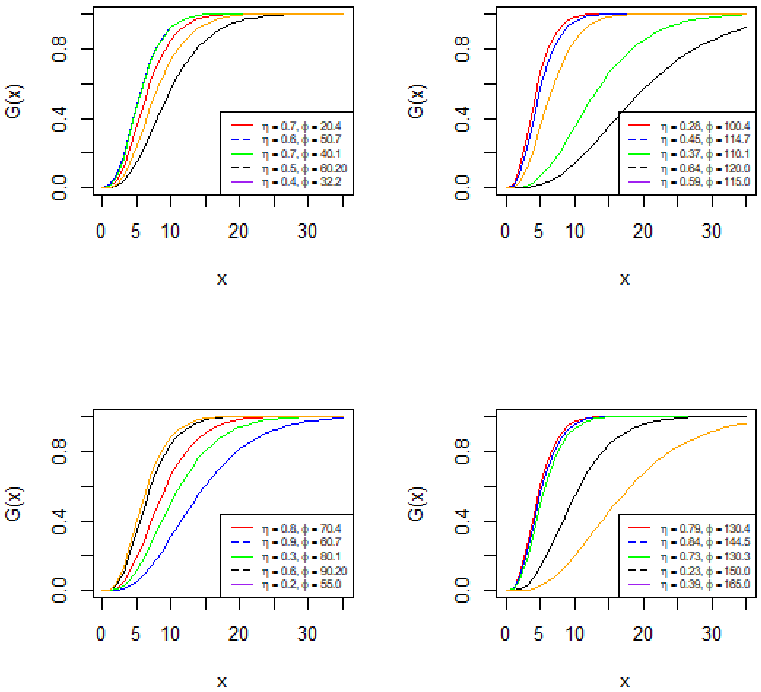

Fig. 2. The graphical plots of the cumulative distribution function (for some selected but different real points of η and ϕ) of a Coupon distribution

The curves displayed in Fig. 1 are not bell-shaped, but are positively skewed, unimodal, and right tailed. In addition, the curve shows that increasing the value of ϕ leads to a considerable increase in the peak of the curve. In addition, the curves displayed in Fig. 2 shows that the cumulative distribution function converges to one.

3 Mathematical properties

In this section, we derive and present some of the mathematical properties of the Copoun distribution (CpD) such as the moment and other related measures, and moment generating function.

3.1 Moments of the copoun distribution

We derive the rth moment of the Copoun distribution (CpD) in this subsection.

Theorem 1

Given a random variable X, following the Copoun distribution (CpD), the kth order moment about origin, $E(X^k)$ of the Copoun distribution is given by

$$\mu_k' = E(X^k) = \frac{\phi\eta^2}{6(\phi+\eta)}\left[\frac{k!}{\eta^{k+1}} + \frac{\phi(k+3)!}{\eta^{k+2}}\right] \tag{3.1}$$

Proof:

The kth crude or uncorrected moments of a random variable and can be written as

$$\mu_k' = E(X^k) = \int_0^\infty x^k g(x;\eta,\phi)dx \tag{3.2}$$

$$= \int_0^\infty x^k \cdot \frac{\eta^2}{(\phi+\eta)}\left[1 + \frac{\phi\eta^2 x^3}{6}\right] e^{-\eta x}\,dx \tag{3.3}$$

$$= \frac{\eta^2}{6(\phi+\eta)}\left[\int_0^\infty x^k e^{-\eta x}\,dx + \phi\eta^2 \int_0^\infty x^{k+3} e^{-\eta x}\,dx\right] \tag{3.4}$$

Let $y = \eta x, \Rightarrow x = \frac{y}{\eta}, \Rightarrow y\eta^{-1}; \therefore\, dx = \eta^{-1}dy$. Substituting, we obtain

$$E(X^k) = \frac{\eta^2}{6(\phi+\eta)}\left[\eta^{-k-1} \int_0^\infty y^k e^{-y}\,dy + \phi\eta^{-k-2} \int_0^\infty y^{k+3} e^{-y}\,dy\right] \tag{3.5}$$

Recall, $\Gamma(\varrho + 1) = \int_0^\infty z^\varrho e^{-z}\,dz$, $\Gamma(\varrho + 1) = \varrho\Gamma(\varrho)$, and $\Gamma(s) = (s - 1)!$. Hence,

$$E(X^k) = \frac{\eta^2}{6(\phi+\eta)}\left[\eta^{-k-1}\,\Gamma(k + 1) + \phi\eta^{-k-2}\,\Gamma(k + 4)\right] \tag{3.6}$$

$$\therefore E(X^k) = \frac{\eta^2}{6(\phi+\eta)}\left[\frac{k!}{\eta^{k+1}} + \frac{\phi(k+3)!}{\eta^{k+2}}\right] \tag{3.7}$$

Which completes the proof.

In particular, the first four (4) moments about the origin of the Copoun distribution is obtained by substituting the values of k= 1, 2, 3,4 as follows;

$$\mu_1' = E(X) = \frac{\eta^3+24\eta^2\phi}{6\eta^3(\phi+\eta)} \tag{3.8}$$

$$\mu_2' = E(X^2) = \frac{2\eta^3+120\eta^2\phi}{6\eta^4(\phi+\eta)} \tag{3.9}$$

$$\mu_3' = E(X^3) = \frac{6\eta^3+720\eta^2\phi}{6\eta^5(\phi+\eta)} \tag{3.10}$$

$$\mu_4' = E(X^4) = \frac{24\eta^3+5040\eta^2\phi}{6\eta^6(\phi+\eta)} \tag{3.11}$$

Furthermore, the first four (4) moments about the mean of the Copoun distribution is obtained by substituting the values of k= 1, 2, 3,4 as follows;

$$\sigma^2 = \mu_2 = \frac{144\eta^2+684\phi\eta+11\phi^2}{36\eta^2(\phi+\eta)^2} \tag{3.12}$$

$$\mu_3 = \frac{91\eta^2+324\eta\phi^2+11718\phi\eta^2+864\phi^3}{108\eta^3(\phi+\eta)^3} \tag{3.13}$$

$$\mu_4 = \frac{1463\eta^4+328536\eta^3\phi+263232\eta^2\phi^2+247104\eta\phi^3+31104\phi^4}{432\eta^4(\phi+\eta)^4} \tag{3.14}$$

The coefficient of variation, coefficient of skewness, coefficient of kurtosis, and index of dispersion of the Copoun distribution (CpD), respectively, are given by

$$C.V = \frac{\sqrt{144\eta^2+684\phi\eta+11\phi^2}}{\eta+24\eta\phi} \tag{3.15}$$

$$\sqrt{\beta_1} = \frac{182\eta^2+648\eta\phi^2+23436\phi\eta^2+1728\phi^3}{(144\eta^2+684\phi\eta+11\phi^2)^{\frac{3}{2}}} \tag{3.16}$$

$$\beta_2 = \frac{4389\eta^4+741312\eta^3\phi+789696\eta^2\phi^2+985608\eta\phi^3+93312\phi^4}{(144\eta^2+684\phi\eta+11\phi^2)^2} \tag{3.17}$$

$$\gamma = \frac{144\eta^3+684\phi\eta^2+11\eta\phi^2}{6(\phi+\eta)(\eta^3+24\eta^2\phi)} \tag{3.18}$$

Table 1. The mean, standard deviation, coefficient of variation, skewness, and kurtosis of the CpD(η, ϕ) with different values of the parameter ϕ when $\eta = 0.5$

ϕ	μ_{CpD}	$\sigma^2{}_{CpD}$	CV_{CpD}	$\sqrt{\beta_1}_{CpD}$	$\beta_{2_{CpD}}$
1.1	5.604167	4.297478	1.227064	0.0018468	5.548694
1.3	5.87037	4.137511	1.178022	0.00148192	6.320148
1.5	6.083333	3.99218	1.137876	0.00122478	7.146762
1.7	6.257576	3.860486	1.104129	0.00103614	8.021531
1.9	6.402778	3.741013	1.075184	0.00089331	8.9395
2	6.4667	3.6854	1.0622	0.0008	9.4135
2.3	6.631	3.5331	1.0277	0.0007	10.8909
2.6	6.7634	3.3996	0.9987	0.0006	12.4449
2.9	6.8725	3.2815	0.9737	0.0005	14.0687
3.2	6.964	3.1763	0.952	0.0005	15.7566
3.4	7.0171	3.1123	0.9390	0.0004	16.9149
3.6	7.0650	3.0526	0.9269	0.0004	18.0981
3.8	7.1085	2.9967	0.9157	0.0004	19.3049
4	7.1481	2.9444	0.9053	0.0004	20.5343

Some values of the mean, standard deviation, coefficient of variation, coefficient skewness and coefficient kurtosis for the Copoun distribution (CpD) are obtained for various values of the parameters and the results are presented in Table 1 and Table 2. Table 1 indicates that the mean and kurtosis are increasing as the values of ϕ is increasing. Also, the values of the standard deviation, coefficients of variation, and skewness are decreasing as the values of ϕ is increasing.

Table 2. The mean, standard deviation, coefficient of variation, skewness, and kurtosis of the CpD(η, ϕ) with different values of the parameter ϕ when $\eta = 1.5$

ϕ	μ_{CpD}	$\sigma^2{}_{CpD}$	CV_{CpD}	$\sqrt{\beta_1}_{CpD}$	β_{2CpD}
1.1	1.192308	1.636205	0.965175	0.000732	3.465918
1.3	1.297619	1.624752	0.920704	0.000619	3.579317
1.5	1.388889	1.609194	0.884789	0.000531	3.713935
1.7	1.46875	1.591168	0.85491	0.000462	3.867292
1.9	1.539216	1.571726	0.829489	0.000406	4.036981
2	1.571429	1.561698	0.818107	0.000382	4.127292
2.3	1.657895	1.531112	0.788139	0.000323	4.417488
2.6	1.731707	1.500572	0.763047	0.000277	4.732954
2.9	1.795455	1.47071	0.741589	0.000241	5.070072
3.2	1.851064	1.441856	0.722929	0.000213	5.426153
3.4	1.8844	1.4233	0.7117	0.0002	5.6730
3.6	1.9150	1.4052	0.7014	0.0002	5.9270
3.8	1.9434	1.3877	0.6918	0.0002	6.1875
4	1.9697	1.3708	0.6829	0.0002	6.4543

3.2 Moment Generating Function

Here, we propose the moment generating function for the Copoun distribution (CpD) on this subsection.

Theorem 2

Given a random variable X, following the Copoun distribution (CpD), the moment generating function of X, $M_X(t)$ of the Copoun distribution is given by

$$\mathrm{M_X(t)} = \mathrm{E}(e^{\mathrm{tx}}) = \sum_{\mathrm{k=0}}^{\infty} \binom{\mathrm{t}}{\eta}^{\mathrm{k}} \left[\frac{\eta+\phi(\mathrm{k}+3)(\mathrm{k}+2)(\mathrm{k}+1)}{6(\phi+\eta)}\right] \tag{3.19}$$

Proof:

The moment generating function of X, $M_X(t)$ can be written as

$$\mathrm{M_X(t)} = \mathrm{E}(e^{\mathrm{tx}}) = \int_0^\infty e^{\mathrm{tx}}\, \mathrm{g(x;\eta,\phi)dx} \tag{3.20}$$

$$= \int_0^\infty e^{\mathrm{tx}} \cdot \frac{\eta^2}{(\phi+\eta)}\left[1 + \frac{\phi\eta^2 \mathrm{x}^3}{6}\right] \mathrm{e}^{-\eta \mathrm{x}}\, \mathrm{dx} \tag{3.21}$$

$$= \frac{\eta^2}{6(\phi+\eta)}\int_0^\infty e^{\mathrm{tx}-\eta \mathrm{x}}\, \mathrm{dx} + \frac{\phi\eta^4}{6(\phi+\eta)}\int_0^\infty \mathrm{x}^3 e^{\mathrm{tx}-\eta \mathrm{x}}\, \mathrm{dx} \tag{3.22}$$

$$= \frac{\eta^2}{6(\phi+\eta)}\int_0^\infty e^{\mathrm{tx}} \cdot e^{\eta \mathrm{x}}\, \mathrm{dx} + \frac{\phi\eta^4}{6(\phi+\eta)}\int_0^\infty \mathrm{x}^3 e^{\mathrm{tx}} e^{\eta \mathrm{x}}\, \mathrm{dx} \tag{3.23}$$

Recall that $e^{\mathrm{tx}} = \sum_{\mathrm{k=0}}^{\infty} \frac{(\mathrm{tx})^{\mathrm{k}}}{\mathrm{k!}}$. Substituting, we obtain

$$= \frac{\eta^2}{6(\phi+\eta)}\sum_{\mathrm{k=0}}^{\infty} \frac{\mathrm{t^k}}{\mathrm{k!}}\int_0^\infty \mathrm{x^k} \cdot e^{\eta \mathrm{x}}\, \mathrm{dx} + \frac{\phi\eta^4}{6(\phi+\eta)}\sum_{\mathrm{k=0}}^{\infty} \frac{\mathrm{t^k}}{\mathrm{k!}}\int_0^\infty \mathrm{x^{k+3}} \cdot e^{\eta \mathrm{x}}\, \mathrm{dx} \tag{3.24}$$

Recall, $\int_0^\infty z^w e^{-\varrho z}\, dz = \frac{\Gamma(w+1)}{\varrho^{w+1}}$ and $\Gamma(w) = (w-1)!$. Hence,

$$= \frac{\eta^2}{6(\phi+\eta)}\sum_{\mathrm{k=0}}^{\infty} \frac{\mathrm{t^k}}{\mathrm{k!}} \cdot \frac{\Gamma(\mathrm{k}+1)}{\eta^{\mathrm{k}+1}} + \frac{\phi\eta^4}{6(\phi+\eta)}\sum_{\mathrm{k=0}}^{\infty} \frac{\mathrm{t^k}}{\mathrm{k!}} \cdot \frac{\Gamma(\mathrm{k}+4)}{\eta^{\mathrm{k}+4}} \tag{3.25}$$

$$= \frac{\eta^2}{6\eta(\phi+\eta)} \sum_{k=0}^{\infty} \binom{t}{\eta}^k \cdot \frac{k!}{k!} + \frac{\phi\eta^4}{6\eta^4(\phi+\eta)} \sum_{k=0}^{\infty} \binom{t}{\eta}^k \cdot \frac{(k+3)!}{k!} \tag{3.26}$$

$$\therefore M_X(t) = E(e^{tx}) = \sum_{k=0}^{\infty} \binom{t}{\eta}^k \left[\frac{\eta+\phi(k+3)(k+2)(k+1)}{6(\phi+\eta)}\right] \tag{3.27}$$

Which completes the proof.

4 Conclusion

This paper proposed a new two-parameter distribution known as the Copoun distribution (CpD). The mathematical properties of the Copoun distribution (CpD) such as the moments and other related measures, and moment generating functions were derived and presented. The properties of the new Copoun distribution showed that it can be used to model lifetime datasets with unimodal, positively skewed, and right tailed properties. Finally, the values of the mean, standard deviation, coefficient of variation, skewness, and kurtosis of the Copoun distribution (CpD) with different values of the parameter ϕ when $\eta = 0.5$ and 1.5 indicate that as the parameter ϕ increases, the distribution is shifting to higher values overall (increasing mean) and narrowing around this increased central tendency (decreasing spread, variation and increasing peakedness).

Competing Interests

Authors have declared that no competing interests exist.

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