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Rational Wiener Index and Rational Schultz Index of Graphs †

Belman Gautham Shenoy [1,2], Raghavendra Ananthapadmanabha [1], Badekara Sooryanarayana [3], Prasanna Poojary [4,*] and Vishu Kumar Mallappa [2]

1 Department of Mathematics, Poornapajna College, Volakadu Road, Udupi 576101, Karnataka, India; gautham.shenoy1996@gmail.com (B.G.S.); raghavendra.bhat.a@gmail.com (R.A.)
2 Department of Mathematics, School of Applied Sciences, REVA University, Bengaluru 560064, Karnataka, India; vishukumarm@reva.edu.in
3 Department of Mathematics, Dr. Ambedkar Institute of Technology, BDA Outer Ring Road, Mallathalli 560056, Karnataka, India; dr_bsnrao.mat@drait.edu.in
4 Department of Mathematics, Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education, Manipal 576104, Karnataka, India
* Correspondence: poojary.prasanna@manipal.edu or poojaryprasanna34@gmail.com
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Abstract: In this research paper, we investigate fundamental graph properties within the context of a simple connected graph denoted as G = (V, E). We introduce the concept of the rational Schultz index. In the context of this paper, our main objective is to calculate the rational Wiener index and rational Schultz index for a specific class of graphs. Our focus lies in the analysis and computation of these indices within this particular graph family.

Keywords: rational distance; rational Wiener index; rational Schultz index; complete bipartite graph; crown graph

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1. Introduction

All the graphs considered in this paper are undirected, simple, finite and connected. For the terms not defined, here we refer to [1,2]. The Wiener index is the first topological index defined by H. Wiener in 1947. He used the distances in the molecular graphs of alkanes to calculate their boiling points [3]. The Wiener index $W(G)$ of a connected graph G is defined as the sum of the distances between all unordered pairs of vertices. More precisely,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v),$$

where $V(G)$ is the vertex set of G, and $d(u,v)$ is the length of a shortest path from u to v. For the related work on the Wiener index, we refer to [4–10].

Rational distance was introduced by Raghavendra et al. in [11]. For any two vertices $u, v \in V(G)$, $d(u/v)$ denotes the rational distance from v to u defined as

$$d(u/v) = \begin{cases} \sum_{u_i \in N[u]} \frac{d(u_i,v)}{deg(u)+1} & \text{if} \quad u \neq v. \\ 0 & \text{if} \quad u = v. \end{cases}$$

where $N[u]$ is the closed neighborhood of u, and $deg(u)$ is the degree of the vertex u.

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the graph G. The rational distance matrix of G, denoted by $D_R(G)$, is a square matrix of order n defined by $D_R(G) = (a_{ij})$, where $a_{i,j} = d(v_j/v_i)$.

Further, the rational Wiener index was introduced by Raghavendra et al. in [12] and is given as

$$W_R(G) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}. \tag{1}$$

The Schultz index is another important topological index. The ordinary (vertex) molecular topological index (MTI) of a molecular graph was introduced by Schultz [13] in 1989, and it was eventually named the Schultz index, defined as

$$S(G) = \sum_{\{u,v\} \subseteq V(G)} (\deg u + \deg v) d(u, v). \tag{2}$$

The Schultz index is defined using the concepts of degree and distances in graphs. For the related work on the Schultz index, we refer to [13–17]. The purpose of this paper is to introduce and study the rational Schultz index akin to the Schultz index defined in 2. We now define formally the rational Schultz index of a graph.

The rational Schultz index of a graph G is defined as

$$S_{C_R}(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} (\deg u + \deg v) d(u/v). \tag{3}$$

Our research is motivated by the desire to determine the value and applicability of the rational Wiener index and rational Schultz index. The rational distance takes the closed neighborhood of the vertex into account rather than taking only the vertex, which will provide a better reflection in the case of biological and sociological applications. Therefore, all the factors influencing it can be considered with the help of rational distance. We are particularly motivated by the observation that the rational Schultz index consistently has lower values than the Schultz index in the majority of cases. Understanding the implications of these lower values holds a lot of promise for furthering our understanding of bond analysis between atoms in a molecule. The relevance of the rational Wiener index and rational Schultz index in the QSPR analysis of a given chemical compound can also be investigated, as detailed in references [18–20].

2. The Rational Wiener Index and Rational Schultz Index of a Complete Bipartite Graph

Throughout this section, let $G(V, E)$ be a complete bipartite graph $K_{m,n}$ with $V = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots, u_m\}$, $V_2 = \{v_1, v_2, \ldots, v_n\}$ and $E = \{xy : x \in V_1 \text{ and } y \in V_2\}$.

Lemma 1. For a star graph $K_{1,n}$, the rational Wiener index is given as

$$W_R(K_{1,n}) = \begin{cases} 1/2 & \text{if } n = 1. \\ \frac{n(3n^2+5n-4)}{4(n+1)} & \text{if } n \geq 2. \end{cases}$$

Proof. For $n = 1$, the result is obvious. Let $n \geq 2$. Then, the following holds:

1. The rational distance from a pendent vertex v_j to the central vertex u_1 is

$$\begin{aligned} d(u_1/v_j) &= \frac{\sum_{w \in N[u_1]} d(w, v_j)}{\deg(u_1)+1} = \frac{d(u_1,v_j) + d(v_j,v_j) + \sum_{i=1, i \neq j}^{n-1} d(v_i, v_j)}{n+1}, \\ &= \frac{1+0+2(n-1)}{n+1} = \frac{2n-1}{n+1}. \end{aligned} \tag{4}$$

2. The rational distance from the central vertex u_1 to a pendent vertex v_j is

$$d(v_j/u_1) = \frac{\sum_{w \in N[v_j]} d(w, u_1)}{deg(v_j) + 1} = \frac{d(u_1, u_1) + d(v_j, u_1)}{1+1} = \frac{0+1}{2} = \frac{1}{2}. \tag{5}$$

3. The rational distance from a pendent vertex v_i to another pendent vertex v_j is

$$d(v_j/v_i) = \frac{\sum_{w \in N[v_j]} d(w, v_i)}{deg(v_j) + 1} = \frac{d(u_1, v_i) + d(v_j, v_i)}{1+1} = \frac{1+2}{2} = \frac{3}{2}. \tag{6}$$

Now, substituting Equations (4)–(6) in Equation (2) gives

$$W_R(K_{1,n}) = \frac{1}{2}[\sum_{j=1}^{n} d(u_1/v_j) + \sum_{j=1}^{n} d(v_j/u_1) + \sum_{j=1}^{n} d(v_j/v_j) + \sum_{j=1}^{n}\sum_{i=1,i\neq j}^{n} d(v_j/v_i)],$$
$$= \frac{1}{2}[\sum_{j=1}^{n} \frac{2n-1}{n+1} + \sum_{j=1}^{n}\frac{1}{2} + \sum_{j=1}^{n} 0 + \sum_{j=1}^{n}\sum_{i=1,i\neq j}^{n}\frac{3}{2}],$$
$$= \frac{1}{2}[\frac{2n-1}{n+1}(n) + \frac{1}{2}(n) + \frac{3}{2}(n)(n-1)],$$
$$= \frac{n(3n^2+5n-4)}{4(n+1)}.$$

Hence, the theorem. □

Lemma 2. *For any $n \in \mathbb{Z}^+$, the rational Schultz index of star graph $K_{1,n}$ is given as*

$$S_{C_R}(K_{1,n}) = \begin{cases} 1 & \text{if} \quad n = 1. \\ \frac{n(11n-7)}{4} & \text{if} \quad n \geq 2. \end{cases}$$

Proof. For $n = 1$, the result is trivial. For $n \geq 2$, Equations (4)–(6) in Equation (3) give

$$S_{C_R}(K_{1,n})) = \frac{1}{2}\sum_{u\in V}\sum_{v\in V}[\deg u + \deg v]d(u/v),$$
$$= \frac{1}{2}\sum_{u\in V_1}\sum_{v\in V_1}[\deg u + \deg v]d(u/v) + \frac{1}{2}\sum_{u\in V_1}\sum_{v\in V_2}[\deg u + \deg v]d(u/v)$$
$$+\frac{1}{2}\sum_{u\in V_2}\sum_{v\in V_1}[\deg u + \deg v]d(u/v) + \frac{1}{2}\sum_{u\in V_2}\sum_{v\in V_2}[\deg u + \deg v]d(u/v),$$
$$= \frac{1}{2}[\deg u_1 + \deg u_1]d(u_1/u_1) + \frac{1}{2}\sum_{j=1}^{n}[\deg u_1 + \deg v_j]d(u_1/v_j)$$
$$+\frac{1}{2}\sum_{j=1}^{n}[\deg v_j + \deg u_1]d(v_j/u_1) + \frac{1}{2}\sum_{i=1}^{n}[\deg v_i + \deg v_i]d(v_i/v_i)$$
$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}[\deg v_i + \deg v_j]d(v_i/v_j),$$
$$= \frac{1}{2}[n+n](0) + \frac{1}{2}\sum_{j=1}^{n}[n+1]\left(\frac{2n-1}{n+1}\right) + \frac{1}{2}\sum_{j=1}^{n-1}[1+n]\left(\frac{1}{2}\right) + \frac{1}{2}\sum_{i=1}^{n}[1+1](0)$$
$$+\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1,j\neq i}^{n-1}(1+1)\left(\frac{3}{2}\right),$$
$$= \frac{1}{2}\left[0 + (2n-1)n + \frac{n}{2}(n+1) + 0 + 3n(n-1)\right] = \frac{n(11n-7)}{4}.$$

□

Theorem 1. *For any integers m, n with $m \leq n$, the following holds:*

$$1.\ W_R(K_{m,n}) = \begin{cases} 0 & \text{if} \quad m = 1, n = 0. \\ 1/2 & \text{if} \quad m = n = 1. \\ \frac{n(3n^2+5n-4)}{4(n+1)} & \text{if} \quad m = 1, n \geq 2. \\ \frac{mn(4mn+m+n-2)+n(n^2-1)(m+2)+m(m^2-1)(n+2)}{2(m+1)(n+1)} & \text{if} \quad m, n \geq 2. \end{cases}$$

$$2.\ S_{C_R}(K_{m,n}) = \begin{cases} 0 & \text{if} \quad m = 1, n = 0. \\ 1 & \text{if} \quad m = n = 1. \\ \frac{n(11n-7)}{4} & \text{if} \quad m = 1, n \geq 2. \\ (m+n)nm\left(\frac{2(n-1)+1}{2(n+1)} + \frac{2(m-1)+1}{2(m+1)}\right) & \text{if} \quad m, n \geq 2. \\ +nm(n-1)\left(\frac{m+2}{m+1}\right) + nm(m-1)\left(\frac{n+2}{n+1}\right) & \end{cases}$$

Proof.

$$
\begin{aligned}
W_R(K_{m,n}) &= \tfrac{1}{2}\textstyle\sum_{u\in V}\sum_{v\in V} d(u/v),\\
&= \tfrac{1}{2}\textstyle\sum_{u\in V_1}\sum_{v\in V_2} d(u/v) + \tfrac{1}{2}\sum_{u\in V_2}\sum_{v\in V_1} d(u/v) + \tfrac{1}{2}\sum_{u\in V_1}\sum_{v\in V_1} d(u/v) + \tfrac{1}{2}\sum_{u\in V_2}\sum_{v\in V_2} d(u/v),\\
&= \tfrac{1}{2}\textstyle\sum_{i=1}^{n}\sum_{j=1}^{m} d(v_i/u_j) + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} d(u_i/v_j) + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} d(v_i/v_j) + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m} d(u_i/u_j),\\
&= \tfrac{1}{2}\Big[nm\Big(\tfrac{2(n-1)+1}{n+1}\Big) + nm\Big(\tfrac{2(m-1)+1}{m+1}\Big) + n(n-1)\big(\tfrac{m+2}{m+1}\big) + m(m-1)\big(\tfrac{n+2}{n+1}\big)\Big].
\end{aligned}
$$

$$
\begin{aligned}
S_{C_R}(K_{m,n}) &= \tfrac{1}{2}\textstyle\sum_{u\in V}\sum_{v\in V}[deg(u)+deg(v)]d(u/v),\\
&= \tfrac{1}{2}\textstyle\sum_{u\in V_1}\sum_{v\in V_2}[deg(u)+deg(v)]d(u/v) + \tfrac{1}{2}\sum_{u\in V_2}\sum_{v\in V_1}[deg(u)+deg(v)]d(u/v)\\
&\quad+\tfrac{1}{2}\textstyle\sum_{u\in V_1}\sum_{v\in V_1}[deg(u)+deg(v)]d(u/v) + \tfrac{1}{2}\sum_{u\in V_2}\sum_{v\in V_2}[deg(u)+deg(v)]d(u/v),\\
&= \tfrac{1}{2}\textstyle\sum_{i=1}^{n}\sum_{j=1}^{m}\big[deg(v_i)+deg(u_j)\big]d(v_i/u_j) + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\big[deg(v_j)+deg(u_i)\big]d(u_i/v_j)\\
&\quad+\tfrac{1}{2}\textstyle\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\big[deg(v_i)+deg(v_j)\big]d(v_i/v_j) + \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\big[deg(u_i)+deg(u_j)\big]d(u_i/u_j),\\
&= (m+n)nm\Big(\tfrac{2(n-1)+1}{2(n+1)}\Big) + (m+n)nm\Big(\tfrac{2(m-1)+1}{2(m+1)}\Big) + nm(n-1)\big(\tfrac{m+2}{m+1}\big) + nm(m-1)\big(\tfrac{n+2}{n+1}\big).
\end{aligned}
$$

□

3. The Rational Wiener Index and Rational Schultz of a Wheel Graph

For $n \geq 3$, the wheel W_n is defined to be the graph $K_1 + C_{n-1}$, with the vertex set as $V(W_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and the edge set as $E(W_n) = \{\{v_1, v_i\} | 2 \leq i \leq n\} \cup \{\{v_i, v_{i+1}\} | 2 \leq i \leq n-1\} \cup \{v_2, v_n\}$.

Theorem 2. *For a wheel graph W_n with $n \geq 3$, the rational Wiener index is given as*

$$
W_R(W_n) = \begin{cases} 2 & \text{if} \quad n = 3. \\ \frac{9}{2} & \text{if} \quad n = 4. \\ 10 & \text{if} \quad n = 5. \\ \frac{415}{24} & \text{if} \quad n = 6. \\ \frac{n-1}{2}\left[\frac{2n-5}{n} + \frac{23}{4} + \frac{7}{4}(n-6)\right] & \text{if} \quad n \geq 7. \end{cases}
$$

Proof. Let W_n be the wheel graph with n-vertices where $n \geq 7$. The vertex set of W_n can be divided into two sets. One is a singleton set containing a central vertex only that is $C = \{v_1\}$, and the other is a set containing all the external vertices and is $P = \{v_i | 2 \leq i \leq n\}$. A spanning cycle C with a vertex set has V(C) = P and an edge set E(C) $\subset E(W_n)$. Then, on the basis of a constructed spanning cycle, we can find three types of subsets for set P by fixing a vertex v_i where $2 \leq i \leq n$ that is $P_{i_1} = \{v | d(v_i, v) = 1\}$, $P_{i_2} = \{v | d(v_i, v) = 2\}$ and $P_{i_3} = \{v | d(v_i, v) \geq 3\}$. We can observe $deg(v_1) = n-1$ and $deg(v_i) = 3, \ \ v_i \in P$.

We observe the follow:

1. The rational distance from a vertex in set P to a vertex in set C would be of the form

$$
\begin{aligned}
d(v_1/v_j) &= \tfrac{\sum_{u_i\in N[v_1]} d(u_i, v_j)}{deg(v_1)+1},\\
&= \tfrac{d(v_1,v_j)+d(v_2,v_j)+\cdots+d(v_n,v_j)}{n-1+1} = \tfrac{3+2(n-4)}{n} = \tfrac{2n-5}{n}
\end{aligned}
$$

2. The rational distance from a vertex in set C to a vertex in set P would be of the form

$$
\begin{aligned}
d(v_i/v_1) &= \tfrac{\sum_{u_i\in N[v_i]} d(u_i, v_i)}{deg(v_i)+1},\\
&= \tfrac{d(v_{i-1},v_1)+d(v_i,v_1)+d(v_{i+1},v_1)+d(v_1,v_1)}{3+1} = \tfrac{1+1+1}{4} = \tfrac{3}{4}.
\end{aligned}
$$

3. The rational distance from a vertex in subset P_{i_1} of P to a vertex v_i in set P would be of the form

$$d(v_i/v_j) = \frac{\sum_{u_i \in N[v_i]} d(u_i,v_j)}{deg(v_i)+1},$$
$$= \frac{d(v_k,v_j)+d(v_i,v_j)+d(v_j,v_j)+d(v_1,v_j)}{3+1}, \text{where } v_k \in P_{1j} \text{ for k} \neq j,$$
$$= \frac{2+1+0+1}{4} = 1.$$

4. The rational distance from a vertex in subset P_{i_2} of P to a vertex v_i in set P would be of the form

$$d(v_i/v_j) = \frac{\sum_{u_i \in N[v_i]} d(u_i,v_j)}{deg(v_i)+1},$$
$$= \frac{d(v_{i-1},v_j)+d(v_i,v_j)+d(v_{i+1},v_j)+d(v_1,v_j)}{3+1} = \frac{2+2+1+1}{4} = \frac{6}{4} = \frac{3}{2}.$$

5. The rational distance from a vertex in subset P_{i_3} of P to a vertex v_i in set P would be of the form

$$d(v_i/v_j) = \frac{\sum_{u_i \in N[v_i]} d(u_i,v_i)}{deg(v_i)+1},$$
$$= \frac{d(v_{i-1},v_j)+d(v_i,v_j)+d(v_{i+1},v_j)+d(v_1,v_j)}{3+1} = \frac{2+2+2+1}{4} = \frac{7}{4}.$$

Then, by definition of the rational Wiener index, we can obtain

$$W_R(W_n) = \frac{1}{2}\sum R_{ij},$$

Substituting all the values along with repetitions in the rational Wiener index expression, we can obtain the rational Wiener index as

$$W_R(W_n) = \frac{1}{2}\left[\frac{2n-5}{n}(n-1) + \frac{3}{4}(n-1) + 2(n-1) + 3(n-1) + \frac{7}{4}(n-6)(n-1)\right],$$
$$= \frac{n-1}{2}\left[\frac{2n-5}{n} + \frac{3}{4} + 2 + 3 + \frac{7}{4}(n-6)\right],$$
$$= \frac{n-1}{2}\left[\frac{2n-5}{n} + \frac{23}{4} + \frac{7}{4}(n-6)\right].$$

□

Theorem 3. *For a wheel graph W_n, the rational Schultz index of W_n is given as*

$$S_{C_R}(W_n) = \begin{cases} 8 & \text{if } n = 3. \\ 27 & \text{if } n = 4. \\ \frac{127}{2} & \text{if } n = 5. \\ \frac{340}{3} & \text{if } n = 6. \\ \frac{1}{2}\Big(\frac{2n-5}{n}(n+2)(n-1) + \frac{3}{4}(n+2)(n-1) & \text{if } n \geq 7. \\ \quad +12(n-1) + 18(n-1) + \frac{21}{2}(n-6)(n-1)\Big) & \end{cases}$$

Proof. Let W_n be the wheel graph with n-vertices. The vertex set of W_n can be divided into two sets. One is a single set containing a central vertex only that is $C = \{v_1\}$, and the other is a set containing all the external vertexes and is $P = \{v_i | 2 \leq i \leq n\}$. A spanning cycle C with a vertex set has V(C) = P and an edge set E(C) $\subset E(W_n)$. Then, on the basis of a constructed spanning cycle, we can find three types of subsets for set P by fixing a vertex v_i where $2 \leq i \leq n$ that is $P_{i_1} = \{v | d(v_i, v) = 1\}$, $P_{i_2} = \{v | d(v_i, v) = 2\}$ and $P_{i_3} = \{v | d(v_i, v) \geq 3\}$. We can observe $deg(v_1) = n-1$ and $deg(v_i) = 3, \ v_i \in P$.

We observe the following:

1. The rational distance from a vertex in set P to a vertex in set C would be of the form

$$d(v_1/v_j) = \frac{\sum_{u_i \in N[v_1]} d(u_i, v_j)}{deg(v_1)+1},$$
$$= \frac{d(v_1,v_j)+d(v_2,v_j)+\cdots+d(v_n,v_j)}{n-1+1} = \frac{3+2(n-4)}{n} = \frac{2n-5}{n}.$$

2. The rational distance from a vertex in set C to a vertex in set P would be of the form

$$d(v_i/v_1) = \frac{\sum_{u_i \in N[v_i]} d(u_i, v_i)}{deg(v_i)+1},$$
$$= \frac{d(v_{i-1},v_1)+d(v_i,v_1)+d(v_{i+1},v_1)+d(v_1,v_1)}{3+1} = \frac{1+1+1}{4} = \frac{3}{4}$$

3. The rational distance from a vertex in subset P_{i_1} of P to a vertex v_i in set P would be of the form

$$d(v_i/v_j) = \frac{\sum_{u_i \in N[v_i]} d(u_i, v_j)}{deg(v_i)+1},$$
$$= \frac{d(v_k,v_j)+d(v_i,v_j)+d(v_j,v_j)+d(v_1,v_j)}{3+1}, \text{ where } v_k \in P_{1j} \text{ for k} \neq j,$$
$$= \frac{2+1+0+1}{4} = 1.$$

4. The rational distance from a vertex in subset P_{i_2} of P to a vertex v_i in set P would be of the form

$$d(v_i/v_j) = \frac{\sum_{u_i \in N[v_i]} d(u_i, v_j)}{deg(v_i)+1},$$
$$= \frac{d(v_{i-1},v_j)+d(v_i,v_j)+d(v_{i+1},v_j)+d(v_1,v_j)}{3+1} = \frac{2+2+1+1}{4} = \frac{6}{4} = \frac{3}{2}.$$

5. The rational distance from a vertex in subset P_{i_3} of P to a vertex v_i in set P would be of the form

$$d(v_i/v_j) = \frac{\sum_{u_i \in N[v_i]} d(u_i, v_i)}{deg(v_i)+1},$$
$$= \frac{d(v_{i-1},v_j)+d(v_i,v_j)+d(v_{i+1},v_j)+d(v_1,v_j)}{3+1} = \frac{2+2+2+1}{4} = \frac{7}{4}.$$

Then, by definition of the rational Schultz index, we can obtain

$$S_{C_R}(W_n) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} (d(u) + d(v)) d(u/v),$$

Substituting all the values along with repetitions in the rational Schultz index expression, we can obtain the rational Schultz index as

$$S_{C_R}(W_n) = \frac{1}{2}\left(\frac{2n-5}{n}(n+2)(n-1) + \frac{3}{4}(n+2)(n-1) + 12(n-1) + 18(n-1) + \frac{21}{2}(n-6)(n-1)\right).$$

□

4. The Rational Wiener Index and Rational Schultz of a Friendship Graph

The friendship graph F_s is a collection of s triangles with a common vertex. It may be also pictured as a wheel with every alternate rim edge removed, with vertex set $V(F_s) = \{v_0, v_1, v_2, \ldots, v_{2s-1}, v_{2s}\}$ and edge set $E(F_s) = \{\{v_0, v_i\} | 1 \leq i \leq 2s\} \cup \{\{v_{2i-1}, v_{2i}\} | 1 \leq i \leq s\}$.

Theorem 4. *For a friendship graph F_s, the rational Wiener index of F_s is given as*

$$W_R(F_s) = 2s\left(\frac{10s^2 + 5s - 6}{3(2s+1)}\right).$$

Proof. Let F_s be the friendship graph with n-vertices. The vertex set of F_s can be divided into two sets. One is a single set containing a central vertex only that is $C = \{v_0\}$, and the other is a set containing all the pendent vertices or external vertices and is $P = \{v_i | 1 \leq i \leq 2s\}$. We can observe $d(v_0) = 2s$ and $d(v_i) = 2\ \forall\, v_i \in P$. Considering each subgraph C_3 of F_s, we can establish a subset of P of the form $P_i = \{v_{2i-1}, v_{2i}\}$ where $1 \leq i \leq s$.

We observe the following:

1. The rational distance between two vertices of subset P_k of P would be of the form

$$d(v_{2k-1}/v_{2k}) = d(v_{2k}/v_{2k-1}) = \frac{2}{3}\ \forall\, k, 1 \leq k \leq s.$$

2. The rational distance from a vertex in subset P_k of P to a vertex in subset P_j of P where $j \neq k$ would be of the form

$$d\big(v_{2j-1}/v_{2k-1}\big) = d\big(v_{2j}/v_{2k}\big) = d\big(v_{2j-1}/v_{2k}\big) = d\big(v_{2j}/v_{2k-1}\big) = \frac{5}{3}\ \forall\, j,k, 1 \leq j,k \leq s \text{ and } j \neq k.$$

3. The rational distance from a vertex in set C to a vertex in subset P_k of P would be of the form

$$d(v_{2k-1}/v_0) = d(v_{2k}/v_0) = \frac{2(n-2)}{n}\ \forall\, k, 1 \leq k \leq s \text{ and where } n \text{ represents the order of graph } F_s.$$

4. The rational distance from a vertex in subset P_k of P to a vertex in C would be of the form

$$d(v_0/v_{2k-1}) = d(v_0/v_{2k}) = \frac{2}{3}\ \forall\, k,\ 1 \leq k \leq s.$$

Then, by definition of the rational Wiener index of the graph, we can obtain

$$W_R(F_s) = \frac{2}{3}(n-1) + \frac{5}{6}((n-1)(n-3)) + \frac{(n-2)}{n}(n-1).$$

This can be further simplified to obtain

$$W_R(F_s) = 2s\left(\frac{10s^2 + 5s - 6}{3(2s+1)}\right).$$

□

Theorem5. *For a friendship graph F_s, the rational Schultz index of F_s is given as*

$$S_{C_R}(F_s) = \frac{2}{6}(n+5)(n-1) + \frac{20}{6}(n-3)(n-1) + \left(\frac{n-2}{n}\right)(n-1)(n+1),$$

where n is the order of F_s and n = 2s + 1.

Proof. Let F_s be the friendship graph with n-vertices. The vertex set of F_s can be divided into two sets. One is a single set containing a central vertex only that is $C = \{v_0\}$, and the other is a set containing all the pendent vertices or external vertices and is $P = \{v_i | 1 \leq i \leq 2s\}$. We can observe $d(v_0) = 2s$ and $d(v_i) = 2\ \forall\, v_i \in P$. Considering each subgraph C_3 of F_s, we can establish a subset of P of the form $P_i = \{v_{2i-1}, v_{2i}\}$ where $1 \leq i \leq s$.

We observe the following:

1. The rational distance between two vertices of subset P_k of P would be of the form

$$d(v_{2k-1}/v_{2k}) = d(v_{2k}/v_{2k-1}) = \frac{2}{3}\ \forall\, k, 1 \leq k \leq s.$$

2. The rational distance from a vertex in subset P_k of P to a vertex in subset P_j of P where $i \neq k$ would be of the form

$$d\left(v_{2j-1}/v_{2k-1}\right) = d\left(v_{2j}/v_{2k}\right) = d\left(v_{2j-1}/v_{2k}\right) = d\left(v_{2j}/v_{2k-1}\right) = \frac{5}{3}\ \forall\ j,k,1 \leq j,k \leq s \text{ and } j \neq k.$$

3. The rational distance from a vertex in set C to a vertex in subset P_k of P would be of the form

$$d(v_{2k-1}/v_0) = d(v_{2k}/v_0) = \frac{2(n-2)}{n}\ \forall\ k,1 \leq k \leq s \text{ and where n represents the order of graph } F_s.$$

4. The rational distance from a vertex in subset P_k of P to a vertex in C would be of the form

$$d(v_0/v_{2k-1}) = d(v_0/v_{2k}) = \frac{2}{3}\ \forall\ k,1 \leq k \leq s.$$

Then, by definition of the rational Schultz index, we can obtain

$$S_{C_R}(F_s) = \frac{2}{6}(n+5)(n-1) + \frac{20}{6}(n-3)(n-1) + \left(\frac{n-2}{n}\right)(n-1)(n+1) \text{ where } n \text{ is the order of } F_s \text{ and } n = 2s+1.$$

□

5. The Rational Wiener Index and Rational Schultz of a Crown Graph

An n-crown graph $C_{n,n}$ on $2n$ vertices, $n \geq 3$, is a simple graph with vertex set $V(C_{n,n}) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and edge set $E(C_{n,n}) = \left\{\{x_i, y_j\} \middle| 1 \leq i,j \leq n \text{ and } i \neq j\right\}$. It can also be considered a subgraph of a complete bipartite graph $K_{n,n}$ with edges of the type $\{x_i, y_i\}$, $1 \leq i \leq n$ being removed.

Theorem 6. *For a crown graph $C_{n,n}$, the rational Wiener index of $C_{n,n}$ is given as*

$$W_R(C_{n,n}) = 3n^2 - n + 1.$$

Proof. Let $C_{n,n}$ be the star with n-vertices. The vertex set of $C_{n,n}$ can be divided into two sets. One set is $C = \{u_i | 1 \leq i \leq n\}$ and other set is $D = \{v_i | 1 \leq i \leq n\}$. We can observe $deg(v_i) = deg(u_i) = n-1, \forall\ u_i \in C$ and $v_i \in D$.

We can observe the following:

1. The rational distance between a vertex in C and a vertex in D with an equal index would be of the form

$$d(u_i/v_i) = d(v_i/u_i) = \frac{2n+1}{n}\ \forall\ i,\ 1 \leq i \leq n.$$

2. The rational distance between a vertex in C and a vertex in D with different indexes would be of the form

$$d\left(u_i/v_j\right) = d\left(v_j/u_i\right) = \frac{1+2(n-2)}{n}\ \forall\ i,j,\ 1 \leq i,j \leq n \text{ and } i \neq j.$$

3. The rational distance between vertices with both vertices in C or both vertices in D would be of the form

$$d\left(u_i/u_j\right) = d\left(v_i/v_j\right) = \frac{n+3}{n}\ \forall\ i,j,\ 1 \leq i,j \leq n \text{ and } i \neq j.$$

Then, by definition of the rational Wiener index, we can obtain

$$W_R(C_{n,n}) = \frac{1}{2}\left[\left(\frac{2n+1}{n}\right)2n + \left(\frac{1+2(n-2)}{n}\right)2n(n-1) + \left(\frac{n+3}{n}\right)2n(n-1)\right].$$

This can be further simplified to obtain

$$W_R(C_{n,n}) = 3n^2 - n + 1.$$

□

Theorem 7. *For a crown graph $C_{n,n}$, the rational Schultz index of $C_{n,n}$ is given as*

$$S_R(C_{n,n}) = 2(n-1)\left(3n^2 - n + 1\right).$$

Proof. Let $C_{n,n}$ be the star with n-vertices. The vertex set of $C_{n,n}$ can be divided into two sets. One set is $C = \{u_i | 1 \leq i \leq n\}$ and other set is $D = \{v_i | 1 \leq i \leq n\}$. We can observe $deg(v_i) = deg(u_i) = n - 1, \forall\, u_i \in C$ and $v_i \in D$.

We can observe the following:

1. The rational distance between a vertex in C and a vertex in D with an equal index would be of the form

$$d(u_i/v_i) = d(v_i/u_i) = \frac{2n+1}{n}\ \forall\, i,\ 1 \leq i \leq n.$$

2. The rational distance between a vertex in C and a vertex in D with different index would be of the form

$$d\left(u_i/v_j\right) = d\left(v_j/u_i\right) = \frac{1 + 2(n-2)}{n}\ \forall\, i, j,\ 1 \leq i, j \leq n \text{ and } i \neq j.$$

3. The rational distance between vertices with both in C or both in D would be of the form

$$d\left(u_i/u_j\right) = d\left(v_i/v_j\right) = \frac{n+3}{n}\ \forall\, i, j,\ 1 \leq i, j \leq n \text{ and } i \neq j.$$

Then, by definition of the rational Schultz index, we can obtain

$$S_{C_R}(C_{n,n}) = 2(n-1)\left(3n^2 - n + 1\right).$$

□

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