axioms

MDPI

Article

A Review Study of Prime Period Perfect Gaussian Integer Sequences

Ho-Hsuan Chang *, Shiqi Guan, Miaowang Zeng and Peiyao Chen

DGUT-CNAM Institute, Dongguan University of Technology, Dongguan 523808, China;
guansq@dgut.edu.cn (S.G.); zengmw@dgut.edu.cn (M.Z.); 3152344350abc@gmail.com (P.C.)
* Correspondence: hhchang0415@gmail.com

Abstract: Prime period sequences can serve as the fundamental tool to construct arbitrary composite period sequences. This is a review study of the prime period perfect Gaussian integer sequence (PGIS). When cyclic group $\{1, 2, \ldots, N-1\}$ can be partitioned into k cosets, where $N = kf + 1$ is an odd prime number, the construction of a degree-(k + 1) PGIS can be derived from either matching the flat magnitude spectrum criterion or making the sequence with ideal periodic autocorrelation function (PACF). This is a systematic approach of prime period $N = kf + 1$ PGIS construction, and is applied to construct PGISs with degrees 1, 2, 3 and 5. However, for degrees larger than 3, matching either the flat magnitude spectrum or achieving the ideal PACF encounters a great challenge of solving a system of nonlinear constraint equations. To deal with this problem, the correlation and convolution operations can be applied upon PGISs of lower degrees to generate new PGISs with a degree of 4 and other higher degrees, e.g., 6, 7, 10, 11, 12, 14, 20 and 21 in this paper. In this convolution-based scheme, both degree and pattern of a PGIS vary and can be indeterminate, which is rather nonsystematic compared with the systematic approach. The combination of systematic and nonsystematic schemes contributes great efficiency for constructing abundant PGISs with various degrees and patterns for the associated applications.

Keywords: degree of sequence; ideal periodic autocorrelation function; perfect sequence; PGIS

MSC: 94A55

Citation: Chang, H.-H.; Guan, S.; Zeng, M.; Chen, P. A Review Study of Prime Period Perfect Gaussian Integer Sequences. *Axioms* **2024**, *13*, 159. https://doi.org/10.3390/axioms13030159

Academic Editor: Mircea Merca

Received: 11 January 2024
Revised: 19 February 2024
Accepted: 22 February 2024
Published: 28 February 2024

1. Introduction

A sequence is said to be perfect if and only if the out-of-phase value of the periodic autocorrelation function (PACF) is equal to zero [1–9]. Perfect sequences (PSs) were widely used in modern communication systems for such applications as real-time channel estimation [3,6,8,9], linear system parameter identification [6], equalization [10], synchronization [1,11–13], peak-to-average power ratio reduction [14,15], and modulation [16–19]. However, it is challenging to implement PSs with real or complex coefficients, which requires more memory than integers, and perfect binary sequences of length $N > 4$ and perfect quadri-phase sequences of length $N > 16$ have yet to be found [20]. A Gaussian integer sequence (GIS) is a sequence with coefficients that are complex numbers $a + bj$, where $j = \sqrt{-1}$ and a and b are integers. The construction of a perfect Gaussian integer sequence (PGIS) has become an important research topic [21–30].

By tracing the construction of PGISs, a general form of even-period PGISs was presented in [21], in which the PGISs were constructed by linearly combining four base sequences or their cyclic shift equivalents using Gaussian integer coefficients of equal magnitude. Yang et al. [22] constructed PGISs of odd prime period p by using cyclotomic classes with respect to the multiplicative group of $GF(p)$. Ma et al. [23] later presented PGISs with a period of $p(p+2)$ based on Whiteman's generalized cyclotomy of order two over $\mathbb{Z}_{p(p+2)}$, where p and $(p+2)$ are twin primes. In [24], Chang et al. introduced the concept of the degree of a sequence and constructed degree-2 and degree-3 PGISs of prime period p. Then, they up-sampled these PGISs by a factor of m and filled them with new coefficients to build degree-3 and degree-4 PGISs of arbitrary composite period $N = mp$. Lee et al. focused on constructing degree-2 PGISs of various periods using two-tuple-balanced sequences and cyclic difference sets, where some PGISs are with long and high-energy efficiency properties [25–28]. Algorithms that could generate PGISs of arbitrary period were developed by Pei et al. [29], and one of these algorithms could be applied to construct degree-5 PGISs of prime period $p \equiv 1 \pmod 4$ by applying the generalized Legendre sequences (GLS). PGISs of period qp with degrees equal to or larger than four were proposed in [30].

With the above mentioned significant results of theoretical PGIS study and matured construction techniques, exploring the application of PGIS has gradually become a new research topic [14,15,31–33]. In [14], a PGIS was applied to orthogonal-frequency division multiplexing (OFDM) systems for peak-to-average power ratio (PAPR) reduction. Subsequently, the PGIS was used to construct the transform matrix for the associated precoded OFDM systems to achieve full frequency diversity and obtain optimal bit-error rate [15]. A code division multiple access (CDMA) scheme based on PGIS, called the PGIS-CDMA system, was developed by Chang [31], where a set of PGISs could substitute and outperform the PN codes (e.g., m-sequences, Gold sequences, Kasami sequences, and bent sequences) in a direct sequence (DS) CDMA system. A new application of the PGIS to cryptography referred to [32], in which a hybrid public/private key cryptography scheme based on the PGIS of period $N = pq$ was proposed. This hybrid cryptosystem can take advantage of public and private-key systems, and it is with implementation simplicity for easy adaptation to an IoT platform. PGISs can also be applied to construct a set of zero circular convolution (ZCC) sequences; ZCC sequences feature the advantage of possessing the desired PACF and the ideal periodic cross-correlation function (PCCF) properties. ZCC sequences can be applied for multi-user channel estimation as well as optimal joint symbol detection and channel estimation [33].

Different from that of binary sequence families, there is no upper bound to the available number of PGISs, and one can construct as many different PGISs as one would expect. We can use the degree, pattern and period as three parameters to uniquely define a PGIS. From the application point of view, the available numbers of degrees and patterns of a set of PGISs are desired when the period of the set of PGISs is fixed. For example, the capacity of a PGIS-CDMA system is determined by the number of degrees and patterns [31]. In addition, prime period PGISs can serve as a fundamental tool for constructing arbitrary composite period PGISs [24]. These two reasons stimulate us to perform a review and thorough study of constructing prime period PGIS from both the degree and pattern points of view.

To construct more degrees and patterns of different PGISs is the goal of this study, for which we conclude and group the construction approaches into two schemes: systematic and nonsystematic. When perfect sequences are constructed by matching the flat magnitude spectrum or the ideal PACF criterion, the pattern and degree of a sequence are determined and known in advance, for example, the construction of degree-2 and degree-3 PGISs of prime period in [24] and the construction of a degree-5 PGIS, adopting from the generalized Legengre sequences (GLSs) by Pei et al. [29]. This is the reason this approach can be called a systematic scheme. In this approach, when the cyclotomic order is greater than three, solving constraint equations by matching flat magnitude spectrum criterion becomes a great challenge. With this aspect, we can apply correlation and convolution operations in this study to construct degree-4 and many other degrees which belong to the set $\{6, 7, 10, 11, 12, 14, 20, 21\}$. However, the degree, as well as the pattern, of a PGIS constructed from taking either a correlation or convolution operation between two or more PGISs might vary and is too complicated to be analyzed systematically, which is a case by case condition. This is rather a nonsystematic scheme of PGIS construction compared with the mentioned systematic scheme. One can apply a systematic scheme to construct lower-degree PGISs, and then these lower-degree PGISs can be applied to construct many other higher-degree PGISs using the nonsystematic scheme. The proposed systematic and nonsystematic schemes can be combined to construct efficiently abundant PGISs with various degrees and patterns for the associated different applications.

The structure of this paper is briefly described. Followed by depicting the properties of PGISs in Section 2, Sections 3–6 present a review study of the systematic construction of general prime period PGISs of degrees 1, 2, 3 and 5, respectively. A new study of PGIS construction by correlation and convolution operations is addressed in Section 7, in which there exist abundant degrees and patterns to those PGISs of particular prime periods, e.g., $N = 2^m - 1$ and $N = \frac{p^m-1}{m-1}$, where p is an odd prime. Conclusions are summarized in Section 8.

2. Definitions and PGIS Properties

2.1. Notations

$\delta[\tau]$ is the Kronecker delta sequence of period N. The boldface character $\mathbf{s}$ denotes a sequence or a vector of period N, which is expressed as $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$, and $\mathbf{s}_{-1}^* = \{s^*[(-n)_N]\}_{n=0}^{N-1}$, where the superscript * and $(\cdot)_N$ stand for complex conjugate and modulo N operation, respectively. Let $\mathbf{s}^{(-m)} = \{s[(n+m)_N]\}_{n=0}^{N-1}$ and $\mathbf{s}^{(m)} = \{s[(n-m)_N]\}_{n=0}^{N-1}$ denote the circular shift of $\mathbf{s}$ to the left and right, respectively, by m places, where $0 \leq m \leq N-1$. A set of N different sequences is expressed as $\{\mathbf{s}_m\}_{m=0}^{N-1}$. $\mathbf{S}_1 \circ \mathbf{S}_2$ denotes the component-wise product between $\mathbf{S}_1$ and $\mathbf{S}_2$.

2.2. Definitions

2.2.1. Degree

The degree of a sequence is defined as the number of distinct non-zero elements within one period of the sequence.

2.2.2. Pattern

The pattern of a sequence is defined as the distribution of non-zero elements within one period of the sequence.

We can demonstrate two degree-6 PGISs of period 31, which have different patterns, as follows:

$$(-3,9,9,2,9,-2,2,8,9,-2,-2,2,2,2,8,-5,9,2,-2,8,-2,2,2,-5,2,8,2,-5,8,-5,-5),$$

and

$$(-9,0,0,1,0,-1,1,-2,0,-1,-1,5,1,5,-2,2,0,1,-1,-2,-1,5,5,2,1,-2,5,2,-2,2,2).$$

Notice that since the sequence and pattern are periodic with period N, sequence $\{s[n]\}$ and its circular shifts $\{s[(n \pm m)_N]\}$ in this paper are considered to have the same pattern, as are both of the sequences $\{cs[n]\}$ and $\{s^*[n]\}$. However, the pattern of sequence $\{s[(-n)_N]\}$ may not be the same as that of $\{s[n]\}$.

2.2.3. Circular Convolution

The circular convolution between $\mathbf{s}_1 = \{s_1[n]\}_{n=0}^{N-1}$ and $\mathbf{s}_2 = \{s_2[n]\}_{n=0}^{N-1}$, denoted by $\mathbf{s}_1 \otimes \mathbf{s}_2$=$\{s_{12}[n]\}_{n=0}^{N-1}$, where $s_{12}[n]$ is the nth component of $\mathbf{s}_1 \otimes \mathbf{s}_2$, is defined as

$$s_{12}[n] = \sum_{\tau=0}^{N-1} s_1[\tau] s_2[(n-\tau)_N],$$

where $(\cdot)_N$ denotes modulo N.

2.2.4. PACF

Let $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$ denote a sequence of period N, where $s[n]$ is the nth component of $\mathbf{s}$. $\mathbf{R}_s \equiv \mathbf{s} \otimes \mathbf{s}_{-1}^* = \{R[\tau]\}_{\tau=0}^{N-1}$ denotes the periodic autocorrelation function (PACF) of $\mathbf{s}$, i.e.,

$$R[\tau] = \sum_{n=0}^{N-1} s[n]s^*[(n-\tau)_N],$$

where $\mathbf{s}_{-1} = \{s[(-n)_N]\}_{n=0}^{N-1}$. Let $\mathbf{S} = \{S[n]\}_{n=0}^{N-1}$ denote the discrete Fourier transform (DFT) of $\mathbf{s}$. Then, the DFT of $\mathbf{R}_s$ is $\mathbf{S} \circ \mathbf{S}^*$=$|\mathbf{S}|^2$, where $|\cdot|$ denotes the Euclidean norm. The sequence $\mathbf{s}$ is called perfect if and only if $\mathbf{R_s} = E \cdot \delta[\tau]$, where E is the energy of the sequence $\mathbf{s}$. The DFT pair relationship between $\mathbf{R_s} = E \cdot \delta[\tau]$ and $\mathbf{S} \circ \mathbf{S}^* = |\mathbf{S}|^2$ indicates that a sequence $\mathbf{s}$ is perfect if and only if the magnitude spectrum of $\mathbf{s}$ is flat, i.e., $|S[n]| = \sqrt{E}$, $\forall\, 0 \le n \le N-1$.

2.2.5. PCCF

The periodic cross-correlation function (PCCF) between $\mathbf{s}_1 = \{s_1[\tau]\}_{\tau=0}^{N-1}$ and $\mathbf{s}_2 = \{s_2[\tau]\}_{\tau=0}^{N-1}$ is defined as

$$R_{1,2}[\tau] = \sum_{n=0}^{N-1} s_1[n]s_2^*[(n-\tau)_N].$$

2.2.6. Coset

Let $N = KM + 1$ be an odd prime number; thus, $\mathbf{Z}_N = \{1, 2, \ldots, N-1\}$ is both a multiplicative group and a cyclic group [34]. If $\alpha \in \mathbf{Z}_N$ is a primitive element, it follows that $\alpha^{N-1} = 1$. Let $\mathbf{H} \equiv \{\alpha^{m \cdot K}\}_{m=0}^{M-1}$ and $\gamma \in \mathbf{Z}_N$. The subset $\mathbf{H}\gamma \equiv \{h\gamma | h \in \mathbf{H}\}$ is called the right coset of subgroup $\mathbf{H}$ generated by γ. Define

$$\mathbf{H}_k \equiv \mathbf{H}\alpha^k,\ k = 0, 1, \ldots, K-1. \tag{1}$$

It is easy to show that $\mathbf{H}_k$, $k = 0, 1, \ldots, K-1$, are distinct right cosets of $\mathbf{H}$, where $\mathbf{H} = \mathbf{H}_0$. It can be further shown that $\mathbf{Z}_N = \mathbf{H}_0 \cup \mathbf{H}_1 \cup \cdots \cup \mathbf{H}_{K-1}$ and $|\mathbf{Z}_N| = |\mathbf{H}_0| + |\mathbf{H}_1| + \cdots + |\mathbf{H}_{K-1}| = KM$. It is noted that $\mathbf{H}_k = \mathbf{H}\alpha^k = \mathbf{H}\alpha^{(k+m \cdot K)}$, where $\alpha^{k+m \cdot K}$, $m = 0, 1, \ldots, M-1$ belong to the same coset.

2.3. PGIS Properties

Only parts of the PGIS properties, which are related to this study, are summarized to form the following theorem. In particular, property 7 is applied for nonsystematic PGIS construction.

Theorem 1. *Let* $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$, $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *be three PGISs of prime period N. The following sequences are also PGISs of period N:*

(1) $\{s[(n \pm m)_N]\}$, *where m is any integer;*
(2) $\{cs[n]\}$, *where c is any nonzero Gaussian integer;*
(3) $\{s^*[n]\}$, *where* $s^*[n]$ *denotes complex conjugation;*
(4) $\{S[k]\}$, *the DFT of* $\{s[n]\}$, *given that* $\{s[n]\}$ *is with a constant amplitude;*
(5) $\{s[(-n)_N]\}$;
(6) $\mathbf{s}_1 \otimes \mathbf{s}_2$;
(7) $\{R_{1,2}[\tau]\}_{\tau=0}^{N-1}$, $\{R_{2,1}[\tau]\}_{\tau=0}^{N-1}$, $\{R_{1,1}[\tau]\}_{\tau=0}^{N-1}$ *and* $\{R_{2,2}[\tau]\}_{\tau=0}^{N-1}$.

Proof. (1). The proof of properties (1) to (6) can refer to [6,31].
(2). To prove the property (7), it is straightforward that $\{s_2^*[(-\tau)_N]\}$ is PGIS, and it has $\{R_{1,2}[\tau]\}$=$\{s_1[\tau]\} \otimes \{s_2^*[(-\tau)_N]\}$. The convolution between $\{s_1[\tau]\}$ and $\{s_2^*[(-\tau)_N]\}$ yields that $\{R_{1,2}[\tau]\}$ is also a PGIS of period N by property (6). Similarly, $\{R_{2,1}[\tau]\}$=$\{s_2[\tau]\} \otimes \{s_1^*[(-\tau)_N]\}$, $\{R_{1,1}[\tau]\}$=$\{s_1[\tau]\} \otimes \{s_1^*[(-\tau)_N]\}$ and $\{R_{2,2}[\tau]\}$=$\{s_2[\tau]\} \otimes \{s_2^*[(-\tau)_N]\}$ are also PGISs of period N as well. □

When the degree of sequence is a great concern, one can apply the cyclotomic class for systematically constructing PS and PSIS according to the following theorem.

Theorem 2. *Let a cyclic group $\mathbf{Z}_N = \{1, 2, \ldots, N-1\}$ be partitioned into K cosets, where each coset contains M elements and $N = 1 + KM$. Let $c_1, c_2, \ldots, c_F$ be all the F positive factors of $N-1$. There exist $F+1$ classes of PSs of period N with degrees* $1, 1 + \frac{N-1}{c_1}, 1 + \frac{N-1}{c_2}, \ldots, 1 + \frac{N-1}{c_F}$*, respectively. It is noted that* $K \in \{\frac{N-1}{c_1}, \frac{N-1}{c_2}, \ldots, \frac{N-1}{c_F}\}$.

Proof. Refer to [35]. □

Consider the case of $N = 13$, where the six positive factors of $N-1 = 12$ are 1, 2, 3, 4, 6 and 12, respectively. Therefore, the corresponding degrees of the PSs or PGISs are given by 13, 7, 5, 4, 3, and 2, respectively. We would like to mention that Theorem 2 can ensure the existence of six patterns of PSs with period $N = 13$; however, it is challenging to construct PGISs of these six patterns, where the coefficients of these sequences should be Gaussian integer numbers.

3. Unique Degree-1 PGIS

To encompass a broader scope of sequence degree, a particular degree-1 PGIS, which is originated from Kronecker delta sequence $\delta[\tau]$, is addressed in this section.

Theorem 3. *For any nonzero Gaussian integer a, sequence* $\mathbf{s} = (a, \underbrace{0, \ldots, 0}_{N-1})$ *and all $N-1$ other circular shifts of* $\mathbf{s}$*, with notation* $\mathbf{s}^{(n)}$*, are the only existing degree-1 PGISs of period N. In other words,* $(a, 0, \ldots, 0)$ *is the unique pattern of degree-1 PGIS.*

Proof. At first, the number of different nonzero elements for degree-1 PGIS is one. The DFT of $\mathbf{s} = (a, \underbrace{0, \ldots, 0}_{N-1})$ is $\mathbf{S} = (a, \underbrace{a, \ldots, a}_{N-1})$, where $\mathbf{S}$ meets the flat magnitude spectrum criterion for sequence $\mathbf{s}$ to be a PGIS. Let $\mathbf{s}^{(n)} = (\underbrace{0, \ldots, 0}_{n}, a, \underbrace{0, \ldots, 0}_{N-n-1})$ be the n-shift of $\mathbf{s}$. The DFT of $\mathbf{s}^{(n)}$ is $\mathbf{S}^{(n)} = \left\{ae^{-j\frac{2\pi nm}{N}}\right\}_{m=0}^{N-1}$, where $\left|ae^{-j\frac{2\pi nm}{N}}\right| = |a|, \forall m$. This infers that $\mathbf{s}^{(n)}$ is a degree-1 PGIS, and this is valid for all $n = 1, \ldots, N-1$.

When there exist two "a" elements in this sequence, e.g., $\mathbf{s} + \mathbf{s}^{(n)}$, the DFT of sequence $\mathbf{s} + \mathbf{s}^{(n)}$ becomes $\mathbf{S} + \mathbf{S}^{(n)} = \left\{a\left(1 + e^{-j\frac{2\pi nm}{N}}\right)\right\}_{m=0}^{N-1}$, where $\left|a(1 + e^{-j\frac{2\pi nm}{N}})\right| \neq \left|a(1 + e^{-j\frac{2\pi nk}{N}})\right|$, $0 \leq m, k \leq N-1, m \neq k$. The flat magnitude spectrum criterion for a sequence to be perfect cannot be maintained in this situation. By extending this result, when sequence exists more than two "a" elements, the DFT of this sequence becomes $\left\{a(1 + \sum_{n \neq 0} e^{-j\frac{2\pi nm}{N}})\right\}_{m=0}^{N-1}$ for some n, which it is straightforward that $\left|a(1 + \sum_{n \neq 0} e^{-j\frac{2\pi nm}{N}})\right| \neq \left|a(1 + \sum_{n \neq 0} e^{-j\frac{2\pi nk}{N}})\right|$ is true, $0 \leq m, k \leq N-1, m \neq k$. This leads to the conclusion that the sequence can no longer be a degree-1 PGIS when there exist two or more "a" elements. □

4. Degree-2 PGISs Construction

In addition to the fact that degree-2 PGISs can be constructed using the cyclotomic class, the same as PGISs of other degrees, many binary sequences, e.g., m-sequences and the cyclic difference set, can also be adopted to construct degree-2 PGISs, where binary sequence construction is rather a matured topic with many construction schemes or algorithms [6,25]. This implies more abundant patterns of degree-2 than other degrees. The significance of the existence of abundant sequence patterns of degree-2 PGISs has the merit that the more numerous the PGISs, the more they can be applied to generate more new PGISs by means of taking the convolution or correlation operation upon themselves. This topic of the convolution technique on PGIS construction is addressed in Section 7 of this paper.

4.1. Construction Using Cyclotomic Class

4.1.1. Cyclotomic Class of Order 1

Let $N = kf + 1$ be an odd prime. When $k = 1$, there is no partition of the cyclic multiplicative group $Z_N = \{1, 2, \ldots, N-1\}$. In this situation, the pattern of degree-2 PGIS is

$$\mathbf{s} = (a, \underbrace{b, \ldots, b}_{N-1}), \tag{2}$$

where a and b are two nonzero Gaussian integers.
The autocorrelation function of sequence $\mathbf{s} = (a, \underbrace{b, \ldots, b}_{N-1})$ is

$$R[\tau] = \begin{cases} |a|^2 + (N-1)|b|^2, \tau = 0 \\ ab^* + ba_1^* + (N-2)|b|^2, \tau \neq 0. \end{cases}$$

The constraint equation $ab^* + ba^* + (N-2)|b|^2 = 0$ is a necessary as well as sufficient condition for sequence $\mathbf{s}$ to be a degree-2 PGIS with nonzero Gaussian integers $a = x_1 + jy_1$ and $b = x_0 + jy_0$. This equation can be further simplified as

$$2(x_0x_1 + y_0y_1) + (N-2)(x_0^2 + y_0^2) = 0. \tag{3}$$

Example 1. *When $f = 4$ and $N = 4 + 1 = 5$, Gaussian integers $a = 9 + 2j$ and $b = -1 - 3j$ fulfill (3). A degree-2 PGIS of period five is given by*

$$\mathbf{s} = (9 + 2j, -1 - 3j, -1 - 3j, -1 - 3j, -1 - 3j).$$

4.1.2. Cyclotomic Class of Order 2

When k = 2 and $N = 2f + 1$ is an odd prime, the cyclic group $Z_N = \{1, 2, \ldots, N-1\}$ can be partitioned into two cosets $Z_N = \mathrm{H}b_0 \cup \mathrm{H}b_1$, where $\alpha^{2f} = 1$, $\mathrm{H}b_0 = \{\alpha^{2n}\}_{n=0}^{f-1}$ and $\mathrm{H}b_1 = \{\alpha^{2n+1}\}_{n=0}^{f-1}$. To construct PGIS, at first, three base sequences $\mathbf{x}_\delta$ and $\mathbf{x}_i = \{x_i[n]\}_{n=0}^{N-1}, i = 0, 1$ are defined as follows:

$$\mathbf{x}_\delta = (1, \underbrace{0, \ldots, 0}_{N-1}),$$

$$x_i[n] = \begin{cases} 1, & n \in \mathrm{H}b_i \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4. *Let $N = 2f + 1$ be an odd prime and f be an odd integer. The sequence $\mathbf{s} = a(\mathbf{x}_\delta + \mathbf{x}_0) + b\mathbf{x}_1$ with two nonzero Gaussian integers a and b is a degree-2 PGIS if the following constraint equation holds:*

$$\begin{aligned} |a(f+1) + bf| &= \left| \frac{(a-b)\left(1 + j\sqrt{N}\right)}{2} \right| \\ &= \left| \frac{(a-b)\left(1 - j\sqrt{N}\right)}{2} \right|. \end{aligned} \tag{4}$$

Proof. Refer to [24]. □

Corollary 1. *Let $N = 2f + 1$ be an odd prime and f be an odd integer. The sequence $\mathbf{s} = b(\mathbf{x}_\delta + \mathbf{x}_1) + a\mathbf{x}_0$ with two nonzero Gaussian integers a and b is a degree-2 PGIS if*

$$\begin{aligned} |b(f+1) + af| &= \left| \frac{(b-a)\left(1 + j\sqrt{N}\right)}{2} \right| \\ &= \left| \frac{(b-a)\left(1 - j\sqrt{N}\right)}{2} \right|. \end{aligned} \tag{5}$$

Proof. The flat magnitude spectrum criterion leads to the constraint of Equation (5). □

Let $a = a_R + ja_I$ and $b = b_R + jb_I$. The constraint Equation (4) infers that the following equation should be fulfilled:

$$\frac{b_R^2 + b_I^2}{(a_R + b_R)^2 + (a_I + b_I)^2} = \frac{f+1}{2}. \tag{6}$$

And equation (5) infers that

$$\frac{a_R^2 + a_I^2}{(a_R + b_R)^2 + (a_I + b_I)^2} = \frac{f+1}{2}. \tag{7}$$

Example 2. *When $f = 15$ and $N = 2f + 1 = 31$, Gaussian integers $a = -5$ and $b = 6 + 2j$ fulfill (6). A degree-2 PGIS of period 31 is given by*

$$\mathbf{s} = (a, a, a, b, a, a, b, a, a, a, a, b, b, b, a, b, a, b, a, a, a, b, b, b, b, a, b, b, a, b, b). \tag{8}$$

Example 3. *Gaussian integers $a = 2 - 6j$ and $b = -3 + 4j$ can fulfill (7). A degree-2 PGIS of period 31 but with a different pattern from that of (8) is*

$$\mathbf{s} = (b, a, a, b, a, a, b, a, a, a, a, b, b, b, a, b, a, b, a, a, a, b, b, b, b, a, b, b, a, b, b). \tag{9}$$

However, there exists no degree-2 PGIS of prime period $N = 2f + 1$ when f is an even integer if the base sequences $\mathbf{x}_\delta$, $\mathbf{x}_0$ and $\mathbf{x}_1$ are applied for sequence construction [24].

4.2. Degree-2 PGISs of Arbitrary Prime Period

Let us define two base sequences $\mathbf{x}_a$, $\mathbf{x}_b$ as follows:

$$\mathbf{x}_a = (1, \underbrace{1, \ldots, 1}_{N-1}), \tag{10}$$

$$\mathbf{x}_b = (N-1, \underbrace{-1, \ldots, -1}_{N-1}),$$

Base sequences $\mathbf{x}_a$ and $\mathbf{x}_b$ can be applied to construct a degree-2 PGIS of prime period $N = 2f + 1$ for both even and odd f according to Theorem 5.

Theorem 5. *The sequence $\mathbf{s} = a\mathbf{x}_a + b\mathbf{x}_b$ with nonzero Gaussian integers a and b is a degree-2 PGIS if $|a| = |b|$.*

Proof. Refer to [24]. □

Above all, there exist three different sequence patterns to degree-2 PGISs of odd prime period $N = 2f + 1$ when f is odd, but there is only one pattern when f is even. However, note that any degree-2 PGISs constructed based on Theorem 5 belong to the same sequence pattern as that of (2). To explain the reason, the two base sequences that span the sequence pattern in (2) are $\mathbf{x}_\delta$ and $\mathbf{x}_c = (0, \underbrace{1, \ldots, 1}_{N-1})$, for which $\{\mathbf{x}_\delta, \mathbf{x}_c\}$ and $\{\mathbf{x}_a, \mathbf{x}_b\}$ can span the same vector space.

From the sequence application point of view, it is desirable to design as many distinct sequences as possible for a given period. There do exist many other sequence patterns in the degree-2 PGIS family of a particular prime period, addressed in the following two subsections.

4.3. Degree-2 PGISs Adopting from Ternary Perfect Sequences

4.3.1. Construction Based on Ternary Perfect Sequences

Ipatov derived a large class of ternary PSs of period $N = \frac{q^m - 1}{q-1}$, where m is an odd number, $q = p^s$, p is an odd prime, and s is an integer [6,36,37]. Having sequence elements that belong to $\{0, +1, -1\}$, the ternary PSs can be adopted to obtain general degree-2 PGISs by replacing $+1$ and -1 with any nonzero Gaussian integers a and $-a$, respectively. The degree-2 PGISs derived from ternary PSs may contain many zero elements. Given $q = 3$ and $m = 3$, the ternary PS of period $13 = \frac{3^3-1}{3-1}$ is $(0,0,1,0,1,1,1,-1,-1,0,1,-1,1)$, and a degree-2 PGIS of period $N = 13$ is given by:

$$\mathbf{s} = (0, 0, a, 0, a, a, a, -a, -a, 0, a, -a, a). \tag{11}$$

4.3.2. Construction Based on CIDTS

The second type of degree-2 PGISs can be built, adopting from the correlation identity-derived ternary sequences (CIDTS) [6]. Momentarily, we will present only the construction of 12 different degree-2 PGISs of prime period $N = 2^5 - 1$ based on CIDTS, which are $\{\mathbf{t}_1, \cdots, \mathbf{t}_{12}\}$, in Table 1. The detailed construction rules of this scheme can refer to Section 7.5.

4.4. Degree-2 PGISs of Prime Period $2^m - 1$

In the case of the prime period $N = 2^m - 1$ family, there exist many sequence patterns of degree-2 PGISs. In [25], Lee et al. constructed four different kinds of degree-2 PGISs of period $N = 2^m - 1$ from the trace representations of Legendre sequences, Hall's sextic residue sequences, m-sequences, and GMW sequences, respectively. Let us present Theorem 6 before addressing the construction of degree-2 PGISs of prime period $N = 2^m - 1$.

Theorem 6. *For any prime number N, the set of quadratic residues of N forms a multiplicative group with cardinality $\frac{N-1}{2}$.*

Proof. The proof of this theorem is omitted here for brevity. □

4.4.1. Degree-2 PGISs from Legendre Sequences

According to Theorem 6, the set of quadratic residues of prime N is isomorphic to the cyclotomic class of order 2. Thus, any degree-2 PGISs of prime period $2^m - 1$ constructed using the trace representations of Legendre sequences belong to the same sequence patterns built according to Theorem 6.

4.4.2. Degree-2 PGISs from Hall's Sextic Residue Sequences

In the case of prime period $N = 4a^2 + 27 = 6f + 1 = 2^m - 1$, where a, f and m are positive integers, e.g., $N = 31$ and $N = 127$, there exist six different sequence patterns of degree-2 PGISs derived from the trace representation of Hall's sextic residue sequences [25].

Table 1. The 23 patterns of degree-2 PGISs of period 31.

PGIS	Sequence Pattern	Coefficients
$\mathbf{s}_1$	$(a,b,b,a,b,a,a,b,b,a,a,a,a,a,b,b,b,a,a,b,a,a,a,b,a,b,a,b,b,b,b)$	$a=-1+3j, b=4j$
$\mathbf{s}_2$	$(a,b,b,b,b,a,b,a,b,a,a,a,b,a,a,b,b,b,a,a,a,a,a,b,b,a,a,b,a,b,b)$	$\mathbf{s}_1$ to $\mathbf{s}_6$
$\mathbf{s}_3$	$(a,b,b,a,b,b,a,a,b,b,b,b,a,b,a,a,b,a,b,a,b,b,b,a,a,a,b,a,a,a,a)$	are derived from
$\mathbf{s}_4$	$(a,a,a,a,a,b,a,a,a,b,b,b,a,b,a,b,a,a,b,a,b,b,b,b,a,a,b,b,a,b,b)$	m-sequences
$\mathbf{s}_5$	$(a,a,a,b,a,b,b,b,a,b,b,a,b,a,b,a,a,b,b,b,b,a,a,a,b,b,a,a,b,a,a)$	
$\mathbf{s}_6$	$(a,a,a,b,a,a,b,b,a,a,a,b,b,b,b,a,a,b,a,b,a,b,b,a,b,b,b,a,b,a,a)$	
$\mathbf{s}_7$	$(a,b,a,a,a,b,a,a,a,a,a,b,a,a,a,a,a,a,a,a,a,a,a,a,b,b,a,b,a,a,a)$	$a=-j, b=1+3j$
$\mathbf{s}_8$	$(b,a,a,b,a,a,b,a,a,a,a,b,b,b,a,b,a,b,a,a,a,b,b,b,b,a,b,b,a,b,b)$	$a=2-6j$,$b=-3+4j$
$\mathbf{s}_9$	$(a,a,a,b,a,a,b,a,a,a,a,b,b,b,a,b,a,b,a,a,a,b,b,b,b,a,b,b,a,b,b)$	$a=-5$,$b=6+2j$
$\mathbf{s}_{10}$	$(a,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b,b)$	$a=88+63j$,$b=-5+j$
$\mathbf{s}_{11}$	$(0,0,a,0,b,a,a,a,b,b,0,a,b,b,0,b,b,a,a,a,a,a,b,a,b,a,0,a,a,b,a)$	a is Gaussian integer and $b=-a$,
$\mathbf{t}_1$	$(b,b,b,a,b,0,a,a,b,0,0,0,a,0,a,0,b,a,0,a,0,0,0,0,a,a,0,0,a,0,0)$	$\mathbf{s}_{11}$ is ternary sequence,
$\mathbf{t}_2$	$(b,0,0,a,0,a,a,0,0,a,a,b,a,b,0,0,0,a,a,0,a,b,b,0,a,0,b,0,0,0,0)$	$\mathbf{t}_1$ to $\mathbf{t}_{12}$ are CIDTS
$\mathbf{t}_3$	$(b,0,0,b,0,a,b,0,0,a,a,a,b,a,0,0,0,b,a,0,a,a,a,0,b,0,a,0,0,0,0)$	constructed based on m-sequences
$\mathbf{t}_4$	$(b,a,a,0,a,0,0,0,a,0,0,a,0,a,0,b,a,0,0,0,0,a,a,b,0,0,a,b,0,b,b)$	
$\mathbf{t}_5$	$(b,0,0,0,0,b,0,a,0,b,b,a,0,a,a,0,0,0,b,a,b,a,a,0,0,a,a,0,a,0,0)$	
$\mathbf{t}_6$	$(b,0,0,a,0,0,a,a,0,0,0,0,a,0,a,b,0,a,0,a,0,0,0,b,a,a,0,b,a,b,b)$	
$\mathbf{t}_7$	$(b,b,b,0,b,a,0,0,b,a,a,0,0,0,0,a,b,0,a,0,a,0,0,a,0,0,0,a,0,a,a)$	
$\mathbf{t}_8$	$(b,0,0,0,0,a,0,b,0,a,a,a,0,a,b,0,0,0,a,b,a,a,a,0,0,b,a,0,b,0,0)$	
$\mathbf{t}_9$	$(b,a,a,0,a,b,0,0,a,b,b,0,0,0,0,a,a,0,b,0,b,0,0,a,0,0,0,a,0,a,a)$	
$\mathbf{t}_{10}$	$(b,0,0,b,0,0,b,a,0,0,0,0,b,0,a,a,0,b,0,a,0,0,0,a,b,a,0,a,a,a,a)$	
$\mathbf{t}_{11}$	$(b,a,a,a,a,0,a,b,a,0,0,0,a,0,b,0,a,a,0,b,0,0,0,0,a,b,0,0,b,0,0)$	
$\mathbf{t}_{12}$	$(b,a,a,0,a,0,0,0,a,0,0,b,0,b,0,a,a,0,0,0,0,b,b,a,0,0,b,a,0,a,a)$	

$\mathbf{s}_{10}$, $\mathbf{s}_8$ and $\mathbf{s}_9$ are constructed using cyclotomic class of orders 1, 2, and 2, respectively, and $\mathbf{s}_7$ is from (12)

4.4.3. Degree-2 PGISs from m-Sequences

In the case of degree-2 PGISs derived from m-sequences of period $2^m - 1$, the number of distinct $m-$sequence is $\frac{\phi(2^m-1)}{m}$, where $\phi(\cdot)$ is *Euler's totient function*. For example, when N = 7, there exist two patterns which are the same as that of the cyclotomic class of order 2. In case of N = 31, the existing six sequence patterns are the same as that based on the Hall's sextic residue sequences [25]. There exist 18 different sequence patterns when N = 127; these patterns are different from both the Hall's sextic residue sequences and the cyclotomic class of order 2. More details on this topic can be referred to in [25].

4.4.4. Degree-2 PGISs from Cyclic Difference Set

TABLE II in [26] presents the cyclic difference sets (ν, κ, λ) of order $(\kappa - \lambda) \leq 30$, among which the two cyclic sets that belong to the family of prime period $N = 2^m - 1$ are $(31, 6, 1)$ and $(31, 15, 7)$. The degree-2 PGIS pattern constructed using $(31, 15, 7)$ belongs to one of six patterns derived from m-sequences of period $2^5 - 1$; while a new pattern can be generated using $(31, 6, 1)$, when $a = -j$ and $b = 1 + 3j$ are applied, the degree-2 PGIS of period $N = 31$ is

$$\mathbf{s} = (a,b,a,a,a,b,a,a,a,a,a,b,a,a,a,a,a,a,a,a,a,a,a,a,b,b,a,b,a,a,a). \tag{12}$$

5. Degree-3 PGISs Construction

5.1. Construction Using Cyclotomic Class of Order 2

Let $N = 2f + 1$ be an odd prime. When f is odd, the autocorrelation function of sequence $\mathbf{s} = a_2\mathbf{x}_\delta + a_0\mathbf{x}_0 + a_1\mathbf{x}_1$ can be expressed as follows:

$$R[\tau] = \begin{cases} |a_2|^2 + f \cdot (|a_0|^2 + |a_1|^2), & \tau = 0 \\ a_2a_1^* + a_0 \sum_{n \in \mathrm{H}b_0} s^*[(n-\tau)_N] + a_1 \sum_{n \in \mathrm{H}b_1} s^*[(n-\tau)_N], & \tau \in \mathrm{H}b_0 \\ a_2a_0^* + a_0 \sum_{n \in \mathrm{H}b_0} s^*[(n-\tau)_N] + a_1 \sum_{n \in \mathrm{H}b_1} s^*[(n-\tau)_N], & \tau \in \mathrm{H}b_1. \end{cases} \tag{13}$$

When f is even, the autocorrelation function becomes

$$R[\tau] = \begin{cases} |a_2|^2 + f \cdot (|a_0|^2 + |a_1|^2), & \tau = 0 \\ a_2a_0^* + a_0 \sum_{n \in \mathrm{H}b_0} s^*[(n-\tau)_N] + a_1 \sum_{n \in \mathrm{H}b_1} s^*[(n-\tau)_N], & \tau \in \mathrm{H}b_0 \\ a_2a_1^* + a_0 \sum_{n \in \mathrm{H}b_0} s^*[(n-\tau)_N] + a_1 \sum_{n \in \mathrm{H}b_1} s^*[(n-\tau)_N], & \tau \in \mathrm{H}b_1. \end{cases} \tag{14}$$

Let $a_i = x_i + jy_i$, $i = 0, 1, 2$ be three nonzero different Gaussian integers. For an odd f, the necessary and sufficient conditions for sequence $\mathbf{s}$, with its autocorrelation function defined in (13), to be a degree-3 PGIS of period $N = 2f + 1$ leads to the following linear system of two equations with variables x_2 and y_2. The same equations as that of (15) are shown in [22,24], where the derivation of (15) in [24] is based on the frequency domain approach:

$$\begin{cases} y_0x_1 - y_1x_0 = y_2(x_1 - x_0) + x_2(y_0 - y_1) \\ -(\Delta + x_0x_1 + y_0y_1) = x_2(x_1 + x_0) + y_2(y_1 + y_0) \end{cases} \tag{15}$$

where $\Delta = \frac{f-1}{2}((x_0 + x_1)^2 + (y_0 + y_1)^2)$. For an even f, the requirement of $\{R[\tau]\}_{\tau=1}^{N-1} = 0$ in (14) leads to the following linear system of two equations with variables x_2 and y_2. Chang et al. derived the same constraint equations as that of (16) in [24]. However, their derivation is from the frequency domain approach:

$$\begin{cases} \frac{(x_1^2 - x_0^2)}{2} + \frac{(y_1^2 - y_0^2)}{2} = x_2(x_1 - x_0) + y_2(y_1 - y_0) \\ \Delta_x + \Delta_y = -Nx_2(x_1 + x_0) - Ny_2(y_1 + y_0) \end{cases} \tag{16}$$

where

$$\Delta_x = (x_0 + x_1)^2 f^2 - x_0x_1 - \frac{(N+1)(x_0 - x_1)^2}{4}$$

and

$$\Delta_y = (y_0 + y_1)^2 f^2 - y_0y_1 - \frac{(N+1)(y_0 - y_1)^2}{4}.$$

In [29], Pei et al. applied Legendre sequence and Gauss sum to construct degree-3 PGISs. This approach is more efficient in deriving the coefficients of sequence to achieve ideal PACF than solving the constraint of Equations (15) and (16). However, as described in Theorem 5, the sequence pattern constructed based on the Legendre sequences is the same as that based on the cyclotomic class of order 2.

5.2. Degree-3 PGISs of Prime Period $2^m - 1$

This section presents more sequence patterns of degree-3 PGIS of prime period $2^m - 1$, which are derived from taking the circular convolution of two degree-2 PGISs. We present 12 illustrative examples to demonstrate the results of circular convolution in Table 2, for which the former 12 patterns are obtained from circular convolution applied to degree-2

PGISs from Table 1, and the bottom row pattern is constructed using cyclotomic class of order 2.

5.3. Construction from Ternary Perfect Sequences

There exists also a degree-3 PGIS constructed from taking circular convolution between ternary PS and degree-2 PGIS with sequence pattern $\mathbf{s} = (a, \underbrace{b, \ldots, b}_{N-1})$. One more degree-3 PGIS example $\mathbf{s} = \mathbf{s}_{10} \otimes \mathbf{s}_{11}$ of period $N = 2^5 - 1$ is present in Table 2.

Table 2. The 14 patterns of degree-3 PGISs of period 31.

PGIS	Sequence Pattern	Coefficients
$\mathbf{s}_1 \otimes \mathbf{s}_3$	$(a,b,b,c,b,c,c,b,b,c,c,a,c,a,b,b,b,c,c,b,c,a,a,b,c,b,a,b,b,b,b)$	$a = 112 - 44j, b = 16 - 16j,$ $c = -80 + 12j$ (all $\mathbf{s}_i$ are from Table 1)
$\mathbf{s}_1 \otimes \mathbf{s}_4$	$(a,a,a,c,a,b,c,c,a,b,b,b,c,b,c,b,a,c,b,c,b,b,b,b,c,c,b,b,c,b,b)$	
$\mathbf{s}_1 \otimes \mathbf{s}_5$	$(a,c,c,b,c,b,b,b,c,b,b,c,b,c,b,a,c,b,b,b,b,c,c,a,b,b,c,a,b,a,a)$	
$\mathbf{s}_1 \otimes \mathbf{s}_6$	$(a,b,b,a,c,b,a,b,b,c,c,c,a,c,b,b,b,a,c,b,c,c,c,b,a,b,c,b,b,b,b)$	
$\mathbf{s}_2 \otimes \mathbf{s}_3$	$(a,b,b,c,b,b,c,c,b,b,b,b,c,b,c,a,b,c,b,c,b,b,b,a,c,c,b,a,c,a,a)$	
$\mathbf{s}_2 \otimes \mathbf{s}_4$	$(a,b,b,b,b,a,b,c,b,a,a,c,b,c,c,b,b,b,a,c,a,c,c,b,b,c,c,b,c,b,b)$	
$\mathbf{s}_2 \otimes \mathbf{s}_5$	$(a,b,b,b,b,c,b,a,b,c,c,c,b,c,a,b,b,b,c,a,c,c,c,b,b,a,c,b,a,b,b)$	
$\mathbf{s}_2 \otimes \mathbf{s}_6$	$(a,a,a,b,a,c,b,b,a,c,c,b,b,b,b,c,a,b,c,b,c,b,b,c,b,b,b,c,b,c,c)$	
$\mathbf{s}_3 \otimes \mathbf{s}_5$	$(a,b,b,a,b,b,a,c,b,b,b,b,a,b,c,c,b,a,b,c,b,b,b,c,a,c,b,c,c,c,c)$	
$\mathbf{s}_3 \otimes \mathbf{s}_6$	$(a,c,c,b,c,a,b,b,c,a,a,b,b,b,b,c,c,b,a,b,a,b,b,c,b,b,b,c,b,c,c)$	
$\mathbf{s}_4 \otimes \mathbf{s}_5$	$(a,c,c,b,c,b,b,b,c,b,b,a,b,a,b,c,c,b,b,b,b,a,a,c,b,b,a,c,b,c,c)$	
$\mathbf{s}_4 \otimes \mathbf{s}_6$	$(a,c,c,c,c,b,c,a,c,b,b,b,c,b,a,b,c,c,b,a,b,b,b,b,c,a,b,b,a,b,b)$	
$\mathbf{s}_{10} \otimes \mathbf{s}_{11}$	$(a,a,b,a,c,b,b,b,c,c,a,b,c,c,a,c,c,b,b,b,b,b,c,b,c,b,a,b,b,c,b)$	$a = -25 - 5j, b = 68 - 67j,$ $c = -118 - 57j$
$\mathbf{s}_{cy}$	$(a,b,b,c,b,b,c,b,b,b,b,c,c,c,b,c,b,c,b,b,b,c,c,c,c,b,c,c,b,c,c)$ (construction using cyclotomic class of order 2)	$a = -5 - 5j, b = 3 + 3j,$ $c = -4 - 4j$

6. Degree-5 PGISs Construction

6.1. PGISs Construction Using GLS

Though the authors in [22] did not mention the degree concept of a sequence, they did make efforts in the construction of the degree-5 PGIS of prime period $N = 4f + 1$, for which, by using the cyclotomic class of order four and depending on either odd or even f, two systems of four equations were derived, respectively. However, it is still in a pending situation to solve these two constraint equations from which to show the existence of a prime period degree-5 PGIS. Pei et al. made a breakthrough of successfully constructing the prime period degree-5 PGIS by adopting the GLS instead of using cyclomotic class of order four, though they did not mention the degree-5 concept either [29]. A more detailed study of constructing degree-5 PGIS by adopting GLS is addressed in this section.
At first, the GLS, denoted by $\mathbf{g} = \{g[n]\}_{n=0}^{N-1}$, is defined [38] as follows:

$$g[n] = \begin{cases} 0, & n = 0, \\ \exp\left[\frac{j2\pi(\mathrm{ind}_h n)}{N-1}\right], & n \neq 0 (\mathrm{mod}\, N). \end{cases} \tag{17}$$

In (17), $\mathrm{ind}_h n$ is the index function defined by

$$h^{\mathrm{ind}_h n} \equiv n (\mathrm{mod}\, N).$$

In a further generalization, a scaling factor, $r = 1, 2, \ldots, N-2$, can be introduced in the definition (17), yielding

$$g[n] = \begin{cases} 0, & n = 0, \\ \exp[\frac{j2\pi r(\text{ind}_h n)}{N-1}], & n \neq 0(\bmod N). \end{cases} \tag{18}$$

Lemma 1. *Let $N = 4f+1$ be a prime number. In (18), when the scaling factor r=f, $g[n] \in \{1, j, -1, -j\}, n \neq 0$.*

Proof. Inserting r=f to (18) proves the result. □

Let $\{G[n]\}_{n=0}^{N-1}$ be the DFT of GLS $\mathbf{g}$.

Lemma 2. *Let $N = 4f+1$ be a prime number. In (18), when the scaling factor r=f, the magnitude spectrum of $\mathbf{g} = \{g[n]\}_{n=0}^{N-1}$ is as follows:*

$$|G[n]| = \begin{cases} 0, & n = 0, \\ \sqrt{N}, & n \neq 0(\bmod N). \end{cases} \tag{19}$$

Proof. Refer to [38]. □

We can adopt the results of Lemmas 1 and 2 and apply base sequence $\mathbf{x_a}$, defined in (10), and GLS $\mathbf{g}$ to bound the coefficients of sequences in Gaussian integers, according to Theorem 7.

Theorem 7. *Let $N = 4f+1$ be a prime number and a a nonzero Gaussian integer. The sequence $\mathbf{s} = a \cdot \mathbf{x_a} + N \cdot \mathbf{g}$ is a degree-5 PGIS of period N given that $|a|^2 = N$.*

Proof. When $|a|^2 = N$, the magnitude spectrum of $a \cdot \mathbf{x_a}$ is $N\sqrt{N}\delta[n]$. By applying the result of Lemmas 1 and 2, it is straightforward that the magnitude spectrum of $\mathbf{s} = a \cdot \mathbf{x_a} + N \cdot \mathbf{g}$ is flat, and $g[n] \in \{1, j, -1, -j\}$ implies that the number of different Gaussian integers that appear in sequence $\mathbf{s}$ is five. This proves that $\mathbf{s} = a \cdot \mathbf{x_a} + N \cdot \mathbf{g}$ is a degree-5 PGIS. □

Examples 4 and 5 present odd and even f examples of degree-5 PGIS of period $N = 4f+1$, respectively.

Example 4. *When $f = 3$, $N = 4 \cdot 3 + 1 = 13$. Let $a = 2 - 3j$, where $|a|^2 = 13$. The GLS $g = (0, 1, j, 1, -1, j, j, -j, -j, 1, -1, -j, -1)$. A degree-5 PGIS $\mathbf{s} = a \cdot \mathbf{x_a} + 13 \cdot \mathbf{g}$ of period 13 is given by*

$$\mathbf{s} = (a, b, c, b, d, c, c, e, e, b, d, e, d), \tag{20}$$

where $a = 2 - 3j$, $b = 15 - 3j$, $c = 2 + 10j$, $d = -11 - 3j$, and $e = 2 - 16j$.

Example 5. *When $f = 4$, $N = 4 \cdot 4 + 1 = 17$. Let $a = 4 + j$, where $|a|^2 = 17$. The GLS $g = (0, 1, -1, j, 1, j, -j, -j, -1, -1, -j, -j, j, 1, j, -1, 1)$. A degree-5 PGIS $\mathbf{s} = a \cdot \mathbf{x_a} + 17 \cdot \mathbf{g}$ of period 17 is given by*

$$\mathbf{s} = (a, b, c, d, b, d, e, e, c, c, e, e, d, b, d, c, b), \tag{21}$$

where $a = 4 + j$, $b = 21 + j$, $c = -13 + j$, $d = 4 + 18j$, and $e = 4 - 16j$.

6.2. Degree-5 PGISs of Prime Period $2^m - 1$

Addressed in the previous section, degree-5 PGIS of arbitrary prime period $N = 4f+1$ can be constructed using the GLS, where for each $N = 4f+1$, there exist two sequence patterns associated with even and odd f, respectively. This section presents the creation of more sequence patterns for the degree-5 PGIS family using the CIDTS scheme [6]. However, this scheme can be applied only to a particular prime period, e.g., $N = 2^m - 1$. The principles of the CIDTS scheme are summarized as follows:

Let $\mathbf{s}_b = \{s_b[n]\}_{n=0}^{N-1}$ and $\mathbf{s}_c = \{s_c[n]\}_{n=0}^{N-1}$ be two sequences with two-valued autocorrelation functions (ACFs), i.e.,

$$R_b[\tau] = \begin{cases} A_b, & \tau = 0, \\ B_b, & n \neq 0 \end{cases}$$

$$R_c[\tau] = \begin{cases} A_c, & \tau = 0, \\ B_c, & n \neq 0 \end{cases}$$

The CCF between $\mathbf{s}_b$ and $\mathbf{s}_c$ is

$$R_{b,c}[\tau] = \sum_{n=0}^{N-1} s_b[n] s_c^*[(n-\tau)_N],$$

The following identity is true for periodic correlation functions

$$\sum_{n=0}^{N-1} R_{b,c}[n] R_{b,c}^*[(n-\tau)_N] = \sum_{n=0}^{N-1} R_b[n] R_c^*[(n-\tau)_N].$$

Let $s_a[n] = R_{b,c}[n]$, then $\mathbf{s}_a = \{s_a[n]\}_{n=0}^{N-1}$ is a periodic sequence with two-valued ACF given by [6]

$$R_a[\tau] = \sum_{n=0}^{N-1} R_{b,c}[n] R_{b,c}^*[(n-\tau)_N] = \begin{cases} A_b A_c + (N-1) B_b B_c, & \tau = 0, \\ A_b B_c + A_c B_b + (N-2) B_b B_c, & n \neq 0. \end{cases} \quad (22)$$

From (22), when both $\mathbf{s}_b$ and $\mathbf{s}_c$ are PSs, then $\mathbf{s}_a$ does too. Otherwise, one can still make an necessary adjustment and make $\mathbf{s}_a$ a PS [6]. The result of (22) can be adopted to construct a degree-5 PGIS of particular prime period, e.g., $N = 2^m - 1$. For the $m-$sequence of period $N = 2^5 - 1 = 31$, the six distinct $m-$sequences are $\{\mathbf{m}_1, \cdots, \mathbf{m}_6\}$, which are obtained from $\{\mathbf{s}_1, \cdots, \mathbf{s}_6\}$, listed in Table 1, after substituting $a = 1$ and $b = -1$, respectively. Let us make an adjustment by setting $s_a[n] = R_{b,c}[n] + 1$ to construct three different degree-5 PGISs $\mathbf{s}_a$, presented in Example 6.

Example 6. *At first, when* $\mathbf{s}_b = \{m_1[n]\}_{n=0}^{N-1}$ *and* $\mathbf{s}_c = \{m_2[n]\}_{n=0}^{N-1}$*, by setting* $\{t_{13}[n]\} = \{\frac{R_{b,c}[n]+1}{4}\}$*, a degree-5 PGIS of period 31 is*

$$\mathbf{t}_{13} = (3,0,0,1,0,-1,1,-2,0,-1,-1,1,1,1,-2,2,0,1,-1,-2,-1,1,1,2,1,-2,1,2,-2,2,2). \quad (23)$$

Secondly, when $\mathbf{s}_b = \{m_3[n]\}_{n=0}^{N-1}$ *and* $\mathbf{s}_c = \{m_4[n]\}_{n=0}^{N-1}$*,* $\{t_{14}[n]\} = \{\frac{R_{b,c}[n]+1}{4}\}$ *obtains*

$$\mathbf{t}_{14} = (3,-2,-2,1,-2,0,1,-1,-2,0,0,2,1,2,-1,1,-2,1,0,-1,0,2,2,1,1,-1,2,1,-1,1,1). \quad (24)$$

Finally, when $\mathbf{s}_b = \{m_5[n]\}_{n=0}^{N-1}$ *and* $\mathbf{s}_c = \{m_6[n]\}_{n=0}^{N-1}$ *are applied, the third PGIS is*

$$\mathbf{t}_{15} = (3,-1,-1,2,-1,-2,2,0,-1,-2,-2,1,2,1,0,1,-1,2,-2,0,-2,1,1,1,2,0,1,1,0,1,1). \quad (25)$$

Since $R_{c,b}[(-n)_N] = R_{b,c}[n]$, when $R_{c,b}[n] \neq R_{b,c}[n]$, setting $t_{-a}[n] = \frac{R_{c,b}[n]+1}{4}$ will generate distinct PSIS, where $\{t_{-a}[n]\} = \{t_a[(-n)_N]\}$. Consequently, there exist three other patterns associated with (23)–(25), respectively.

7. PGISs Construction from Convolution and Correlation Operations

Basically there are three parts in this section. The first part consists of Sections 7.1 and 7.2, which addresses the relationship between the circulant matrix and circular convolution, and explores some properties of PGIS construction from convolution. Applying cascading convolution to successfully construct the degree-4 PGIS is discussed in Section 7.3. The last part presents more higher-degree PGIS construction of different types, which includes Sections 7.4–7.6.

7.1. Relationship between Convolution and Circulant Matrix

Let us define a circulant matrix $\mathbf{C}$ of size $N \times N$ based on sequence $\mathbf{c} = \{c[n]\}_{n=0}^{N-1}$, where the elements of $\mathbf{c}$ form the first row of $\mathbf{C}$. With this definition, $\mathbf{C} = \{c[(k-n)_N]\}$, where the (n,k) entry of $\mathbf{C}$, denoted by $C_{n,k}$, is

$$C_{n,k} = c[(k-n)_N].$$

The eigenvalues of a circulant matrix comprise the DFT of the first row of the circulant matrix, and conversely, the first row of a circulant matrix is the inverse DFT of the eigenvalues. In particular, all circulant matrices share the same eigenvectors ([39] and p. 267 [40])

$$\mathbf{y}_m = \frac{1}{\sqrt{N}}[1 \;\; e^{-j2\pi m/N} \;\; \cdots \;\; e^{-j2\pi m(N-1)/N}]^T, \;\; m = 0, 1, \ldots, N-1, \tag{26}$$

where $[\cdot]^T$ denotes the transpose. Let $\mathbf{U}$ be the matrix consisting of the eigenvectors $\mathbf{y}_m$ as columns in order, and Ψ = diag(ψ_k) is the diagonal matrix with diagonal elements $\psi_0, \psi_1, \cdots, \psi_{N-1}$. It is true that $\mathbf{U}\mathbf{U}^H$ = $\mathbf{U}^H\mathbf{U} = \mathbf{I}_N$, where $\mathbf{I}_N$ is an identity matrix.

Lemma 3. *Let $\boldsymbol{C} = \{c[(k-n)_N]\}$ and $\boldsymbol{B} = \{b[(k-n)_N]\}$ be circulant $N \times N$ matrices with eigenvalues ψ_m and β_m, respectively, $m = 0, 1, \ldots, N-1$, where*

$$\psi_m = \sum_{k=0}^{N-1} c[k] e^{-j2\pi km/N},$$

$$\beta_m = \sum_{k=0}^{N-1} b[k] e^{-j2\pi km/N}.$$

Then, $\boldsymbol{C}$ and $\boldsymbol{B}$ commute and

$$\boldsymbol{CB} = \boldsymbol{BC} = \boldsymbol{U}\Omega\boldsymbol{U}^H,$$

where Ω=diag($\psi_m\beta_m$) is the diagonal matrix with diagonal elements $\psi_0\beta_0, \psi_1\beta_1, \cdots, \psi_{N-1}\beta_{N-1}$, $[\cdot]^H$ denotes the transpose and conjugate operation, and $\boldsymbol{CB}$ is also a circulant matrix.

Proof. Refer to [39,40]. □

Theorem 8. *Let $s_1, s_2, \cdots, s_k$ be k distinct PGISs of period N. Then, $\boldsymbol{s} = \boldsymbol{s}_1 \otimes \boldsymbol{s}_2 \otimes \cdots \otimes \boldsymbol{s}_k$ is a PGIS of period N, where $\otimes$ denotes circular convolution. In addition, $\boldsymbol{s}$ is also a PGIS of period N, when any numbers of $\boldsymbol{s}_i$ are substituted by $\boldsymbol{s}_{-i} = \{s_i[(-n)_N]\}$ or $\boldsymbol{s}^*_{-i}$, $i = 1, \ldots, k$.*

Proof. At first, taking convolution upon two PGISs $\mathbf{s}_1 \otimes \mathbf{s}_2$ obtains a new PGIS, then the resultant PGIS can be convoluted with the third PGIS $\mathbf{s}_3$ to generate other new PGIS, etc. This leads to $\mathbf{s}$, a PGIS of period N. Next, when $\mathbf{s}_i$ is a PGIS, both $\mathbf{s}_{-i}$ and $\mathbf{s}^*_{-i}$ are PGISs as well. This leads to $\mathbf{s}$ also being a PGIS of period N if $\mathbf{s}_i$ is substituted by $\mathbf{s}_{-i}$ or $\mathbf{s}^*_{-i}$. □

With the defined circulant matrix $\mathbf{C}_{s_2} = \{s_2[(n-k)_N]\}$, which is formed based on sequence $\mathbf{s}_{-2} = \{s_2[(-n)_N]\}_{n=0}^{N-1}$, the evaluation of the circular convolution between $\mathbf{s}_1$ and $\mathbf{s}_2$, denoted by $\mathbf{s} = \mathbf{s}_1 \otimes \mathbf{s}_2$, can be obtained by taking the matrix multiplication operation $\mathbf{S} = \mathbf{C}_{s_2}\mathbf{S}_1$ instead, where $\mathbf{S}_1 = [s_1[0] \;\; s_1[1] \cdots s_1[N-1]]^T$ is a $N \times 1$ vector consisting of N

elements from $\mathbf{s}_1 = \{s_1[n]\}_{n=0}^{N-1}$. That is, the values of N components of PGIS $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$ can be derived from the N elements of a $N \times 1$ vector $\mathbf{S} = \mathbf{C}_{s_2}\mathbf{S}_1$.

When $\mathbf{s} = \mathbf{s}_1 \otimes \mathbf{s}_2 \otimes \cdots \otimes \mathbf{s}_r$, $\mathbf{s}$ can be derived from $\mathbf{S} = \mathbf{C}_a\mathbf{S}_1$. In this expression, circulant matrix $\mathbf{C}_a = \mathbf{C}_{s_2}\mathbf{C}_{s_3}\cdots\mathbf{C}_{s_r} = \mathbf{U}\Omega\mathbf{U}^H$ and Ω=diag(ψ_m) is a diagonal matrix with diagonal elements $\psi_0, \psi_1, \cdots, \psi_{N-1}$, where each eigenvalue $\psi_m = \psi_{m2}\psi_{m3}\cdots\psi_{mr}$ is obtained from the product of eigenvalues ψ_{ml} of circulant matrices $\mathbf{C}_{s_l} = \{s_l[(n-k)_N]\}, l = 2, 3, \ldots, r$, respectively. The properties of circulant matrix $\mathbf{C}_a$ may bring insight to determine the degree and pattern of PSIS $\mathbf{s}$ generated from convoluting many PGISs.

7.2. Effect of Convolution on Degree and Pattern Expansion

This section addresses the effectiveness of convolution operation upon two sequences; it can increase the degree and create new pattern to the resultant sequence. This property is described in Theorem 10. The derivation of Theorem 10 is based on Theorem 9 and Lemmas 4 and 5.

Let $\mathrm{H}b_0 = \{\alpha^{kn}\}_{n=0}^{f-1}$ be a subgroup of cyclic group $Z_N = \{1, 2, \ldots, N-1\}$ and $b_i \in Z_N$, where $N = fk+1$. The subset $\mathrm{H}b_i = \{ub_i | u \in \mathrm{H}b_0\}$ is called the right coset of subgroup $\mathrm{H}b_0$ generated by b_i. Let $\mathrm{H}b_0, \mathrm{H}b_1, \ldots, \mathrm{H}b_{k-1}$ be the distinct right cosets of $\mathrm{H}b_0$ in Z_N. Then, $Z_N = \mathrm{H}b_0 \cup \mathrm{H}b_1 \cup \cdots \cup \mathrm{H}b_{k-1}$, which is a disjoint union and $|Z_N| = |\mathrm{H}b_0| + |\mathrm{H}b_1| + \cdots + |\mathrm{H}b_{k-1}| = |\mathrm{H}b_0| + |\mathrm{H}b_0| + \cdots + |\mathrm{H}b_0| = k|\mathrm{H}b_0| = kf$.

Lemma 4. *Let $l, n \in Z_N$, which $l \neq n$. $\sum_{m\in \mathrm{H}b_i} e^{-j2\pi mn/N} = \sum_{m\in \mathrm{H}b_i} e^{-j2\pi ml/N} \Leftrightarrow l, n \in \mathrm{H}b_a$, where $\mathrm{H}b_a \subset \{\mathrm{H}b_0, \mathrm{H}b_1, \ldots, \mathrm{H}b_{k-1}\}$.*

Proof. Let $\mathrm{H}b_0 n = \{un | u \in \mathrm{H}b_0\}$ and $\mathrm{H}b_0 l = \{ul | u \in \mathrm{H}b_0\}$ be two cosets of $\mathrm{H}b_0$ generated by n and l, respectively. If l and n belong to the same coset, which means $\{ul | u \in \mathrm{H}b_0\} = \{un | u \in \mathrm{H}b_0\}$, then $ml \in \{b_i lu | u \in \mathrm{H}b_0\}$ and $mn \in \{b_i lu | u \in \mathrm{H}b_0\}$. This implies that ml and mn belong to the same coset of $\mathrm{H}b_0$ generated by $b_i l$, denoted as $\mathrm{H}b_i l$, where $\mathrm{H}b_i l \subset \{\mathrm{H}b_0, \mathrm{H}b_1, \ldots, \mathrm{H}b_{k-1}\}$. The summation of $e^{-j2\pi mn/N}$ with respect to m, where m comes across the domain of one coset, results in $\sum_{m\in \mathrm{H}b_i} e^{-j2\pi mn/N} = \sum_{m\in \mathrm{H}b_i} e^{-j2\pi ml/N} = \sum_{m\in \mathrm{H}b_i l} e^{-j2\pi m/N}$. Conversely, when $l, m, n \in Z_N$, it is obvious that $\gcd(mn, N) = 1$ and $\gcd(ml, N) = 1$. Since both $e^{-j2\pi ml/N}$ and $e^{-j2\pi mn/N} \in U_N$, where $U_N = \{e^{-j2\pi m/N} | m = 0, 1, \ldots, N-1\}$ denotes the group of Nth roots of unity, $l \neq n \Leftrightarrow e^{-j2\pi ml/N} \neq e^{-j2\pi mn/N}$ and $\sum_{m\in \mathrm{H}b_i} e^{-j2\pi ml/N} = \sum_{m\in \mathrm{H}b_i} e^{-j2\pi mn/N} \Rightarrow \{ml(\mathrm{mod} N) | m \in \mathrm{H}b_i\} = \{mn(\mathrm{mod} N) | m \in \mathrm{H}b_i\}$. This infers that l and n belong to the same coset. $\square$

Let $N = kf + 1$=$k' f' + 1$ be an odd prime. The cyclic group $Z_N = \{1, 2, \ldots, N-1\}$ can be partitioned either into k cosets $\mathrm{H}b_i, i = 0, \ldots, k-1$, or k' cosets $\mathrm{H}b'_i, i = 0, \ldots, k'-1$, respectively, where both $\mathrm{H}b_0 = \{\alpha^{kn}\}_{n=0}^{f-1}$ and $\mathrm{H}b'_0 = \{\alpha^{k'n}\}_{n=0}^{f'-1}$ are subgroups of Z_N^*, $\mathrm{H}b_i$ = $\alpha^i \mathrm{H}b_0$ = $\{\alpha^{kn+i}\}_{n=0}^{f-1}$, $\mathrm{H}b'_i$ = $\alpha^i \mathrm{H}b'_0$ = $\{\alpha^{k'n+i}\}_{n=0}^{f'-1}$, and α is the generator of Z_N. When $k' = mk$ and $m \geq 2$ is an integer, each $\mathrm{H}b_i, i = 0, \ldots, k-1$, can be further partitioned into m cosets, e.g., $\mathrm{H}b_i$ = $\mathrm{H}b'_i \cup \mathrm{H}b'_{k+i} \cup \cdots \cup \mathrm{H}b'_{(m-1)k+i}$, $i = 0, \ldots, k-1$, where the cardinality of all $\mathrm{H}b_i, i = 0, \ldots, k-1$, is f, and that of $\mathrm{H}b'_i, i = 0, \ldots, (mk-1)$, is $f' = f/m$.

Let us define two sequence sets $\mathbf{x}_i = \{x_i[n]\}_{n=0}^{N-1}, i = 0, \ldots, k-1$, and $\mathbf{x}'_i = \{x'_i[n]\}_{n=0}^{N-1}$, $i = 0, \ldots, k'-1$, as follows:

$$x_i[n] = \begin{cases} 1, & n \in \mathrm{H}b_i \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

$$x'_i[n] = \begin{cases} 1, & n \in \mathrm{H}b'_i \\ 0, & \text{otherwise.} \end{cases} \quad (28)$$

The DFTs of $\mathbf{x}_i$ and $\mathbf{x}'_i$ are $\mathbf{X}_i = \{X_i[n]\}_{n=0}^{N-1}$ and $\mathbf{X}'_i = \{X'_i[n]\}_{n=0}^{N-1}$, respectively.

Theorem 9. *All $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_{k-1}$ are $(k+1)$-valued, where the elements of these vectors belong to the following set:*

$$\left\{ f, \sum_{m \in \mathrm{H}b_0} e^{-j2\pi m/N}, \sum_{m \in \mathrm{H}b_1} e^{-j2\pi m/N}, \ldots, \sum_{m \in \mathrm{H}b_{k-1}} e^{-j2\pi m/N} \right\}.$$

Proof. Since $x_i[n] = 1, n \in \mathrm{H}b_i$ and $x_i[n] = 0, n \notin \mathrm{H}b_i$, the nth element of $\mathbf{X}_i = \{X_i[n]\}_{n=0}^{N-1}$ is $X_i[n] = \sum_{m=0}^{N-1} x_i[m] e^{-j2\pi mn/N} = \sum_{m \in \mathrm{H}b_i} e^{-j2\pi mn/N}$. When $n = 0$,

$$X_i[0] = \sum_{m \in \mathrm{H}b_i} e^{-j2\pi mn/N}|_{n=0} = |\mathrm{H}b_i| = f.$$

Given that $m \in \mathrm{H}b_i$, it has $mZ_N = \{m, 2m, \ldots, (N-1)m\}$ and $mZ_N(\mathrm{mod} N) = Z_N$, In other words, $mZ_N = Z_N$ modulo N. Both mZ_N and Z_N have the same partition, which means $mZ_N = \{\mathrm{H}b_0 \cup \mathrm{H}b_1 \cup \cdots \cup \mathrm{H}b_{k-1}\}$ modulo N. Based on the partition of mZ_N, the set $\{X_i[n]\}_{n=1}^{N-1}$ can be grouped into k subsets, i.e.,

$$\{X_i[n]\}_{n=1}^{N-1} = \{X_i[n]\}_{n \in \mathrm{H}b_0}^{N-1} \cup \{X_i[n]\}_{n \in \mathrm{H}b_1}^{N-1} \cup \cdots \cup \{X_i[n]\}_{n \in \mathrm{H}b_{k-1}}^{N-1}.$$

According to Lemma 4, $\{X_i[n]\}_{n \in \mathrm{H}b_d}^{N-1} = \{X_i[m]\}_{m \in \mathrm{H}b_d}^{N-1}, d = 0, 1, \ldots, k-1$. This concludes that for $i = 0, 1, \ldots, k-1$, all $\mathbf{X}_i = \{X_i[n]\}_{n=0}^{N-1}$ are $(k+1)$-valued, which draw distinct $k+1$ values from the following set

$$\left\{ f, \sum_{m \in \mathrm{H}b_0} e^{-j2\pi m/N}, \sum_{m \in \mathrm{H}b_1} e^{-j2\pi m/N}, \ldots, \sum_{m \in \mathrm{H}b_{k-1}} e^{-j2\pi m/N} \right\}.$$

□

Let $N = kf + 1 = k'f' + 1$ be an odd prime, where $k' = mk$ and $m \geq 2$. The relationship between the DFTs of sequences defined in (27) and (28), which are $\mathbf{X}_i = \{X_i[n]\}_{n=0}^{N-1}$ and $\mathbf{X}'_i = \{X'_i[n]\}_{n=0}^{N-1}$, respectively, is governed by the following lemma.

Lemma 5. *$\mathbf{X}_i = \mathbf{X}'_i + \mathbf{X}'_{k+i} + \cdots + \mathbf{X}'_{(m-1)k+i}$, for all $i = 0, \ldots, k-1$. In these vectors, $X_i[0] = mX'_i[0]$, and all elements in set $\{X_i[n]\}_{n \in \mathrm{H}b_i}$ are the same; however, the elements in set $\{X'_i[n]\}_{n \in \mathrm{H}b_i}$ have m different values, which $X_i[n] = X'_i[n] + X'_{k+i}[n] + \cdots + X'_{(m-1)k+i}[n]$, $n \in \mathrm{H}b_i$, for all $i = 0, \ldots, k-1$.*

Proof. Since $\mathrm{H}b_i = \mathrm{H}b'_i \cup \mathrm{H}b'_{k+i} \cup \cdots \cup \mathrm{H}b'_{(m-1)k+i}$, it results in $\mathbf{x}_i = \mathbf{x}'_i + \mathbf{x}'_{k+i} + \cdots + \mathbf{x}'_{(m-1)k+i}$ and derives that $\mathbf{X}_i = \mathbf{X}'_i + \mathbf{X}'_{k+i} + \cdots + \mathbf{X}'_{(m-1)k+i}$ is true, for $i = 0, \ldots, k-1$. By Theorem 9, it is straightforward that $X_i[0] = mX'_i[0]$ and $X_i[n] = X'_i[n] + X'_{k+i}[n] + \cdots + X'_{(m-1)k+i}[n]$, $n \in \mathrm{H}b_i$. □

Let $N = kf + 1 = k'f' + 1$ be an odd prime, where $k' = mk$ and $m \geq 2$. Let $\mathbf{s}_k$ and $\mathbf{s}_{k'}$ be degree-$(k+1)$ and degree-$(k'+1)$ PGISs constructed using sequences $\{\mathbf{x}_i\}_{i=0}^{k}$ and $\{\mathbf{x}'_i\}_{i=0}^{k'}$, respectively. The following theorem can be derived based on the results of Lemma 4 and Lemma 5.

Theorem 10. *The degree and pattern of sequence $\mathbf{s} = \mathbf{s}_k \otimes \mathbf{s}_{k'}$ are the same as those of $\mathbf{s}$. However, when k' and k are relatively coprime, sequence $\mathbf{s} = \mathbf{s}_k \otimes \mathbf{s}_{k'}$ has a new pattern and the degree of PGIS $\mathbf{s}$ is larger than that of $\mathbf{s}_k$ and $\mathbf{s}_{k'}$.*

Proof. Let the DFTs of $\mathbf{s}_k$ and $\mathbf{s}_{k'}$ be $\{X[n]\}$ and $\{X'[n]\}$, respectively. The DFT of $\mathbf{s}_k \otimes \mathbf{s}_{k'}$ is the component-wise product between $\{X[n]\}$ and $\{X'[n]\}$. Based on Lemma 4 and Lemma 5, when $k' = mk$, $\mathrm{H}b'_i \subset \mathrm{H}b_i$, the sequence pattern of $\mathbf{s}_k \otimes \mathbf{s}_{k'}$ is governed by $\mathbf{s}_{k'}$ because all elements in set $\{X_i[n]\}_{n\in \mathrm{H}b_i}$ are the same, but the elements in set $\{X'_i[n]\}_{n\in \mathrm{H}b_i}$ have m different values. When sequences are constructed using base sequences $\{\mathbf{x}_i\}_{i=0}^{k}$ and $\{\mathbf{x}'_i\}_{i=0}^{k'}$, the number of distinct elements of their DFTs determines the degree of the associated sequences according to Theorem 9. This is the reason the degree of $\mathbf{s} = \mathbf{s}_k \otimes \mathbf{s}_{k'}$ is determined also by $\mathbf{s}_{k'}$.

When k' and k are relatively coprime, $\mathrm{H}b'_i \not\subseteq \mathrm{H}b_i$, there exist different non-overlap components between $\{X[n]\}$ and $\{X'[n]\}$. In the case of existing distinct non-overlap components between $\{X[n]\}$ and $\{X'[n]\}$, $\mathbf{s}_k \otimes \mathbf{s}_{k'}$ constructs a new sequence pattern. Moreover, since both elements of $\{X[n]\}$ and $\{X'[n]\}$ are not zeros, the component-wise product between $\{X[n]\}$ and $\{X'[n]\}$ creates only nonzero elements as well, and the number of distinct elements from the component-wise product between $\{X[n]\}$ and $\{X'[n]\}$ is larger than that of both $\{X[n]\}$ and $\{X'[n]\}$. This derives that the degree of $\mathbf{s}_k \otimes \mathbf{s}_{k'}$ is larger than both the $\mathbf{s}_k$ and $\mathbf{s}_{k'}$ sequences. □

7.3. Degree-4 PGISs Construction from Convolution

This section presents the construction of degree-4 PGIS of particular prime period $N = \frac{3^3-1}{3-1} = 13$ and $N = 2^5 - 1 = 31$ from convolution operation. First, let us define three PGISs of period $N = 13$ as follows:

$$\mathbf{s}_t = (0,0,1,0,1,1,1,-1,-1,0,1,-1,1),$$

$$\mathbf{s}_b = (a, \underbrace{b,\ldots,b}_{12}),$$

$$\mathbf{s}_s = (c,d,e,d,d,e,e,e,e,d,d,e,d),$$

where $a = 1+2j$, $b = -2+j$, $c = 5+5j$, $d = 10-6j$ and $e = -6+10j$.

Example 7. *Sequence $\mathbf{s}$=$\mathbf{s}_t \otimes \mathbf{s}_b \otimes \mathbf{s}_s$ is a degree-4 PGIS of period $N = 13$, which is given by*

$$\mathbf{s} = (a,a,b,a,c,b,b,d,d,a,c,d,c), \tag{29}$$

where $a = 684+198j$, $b = 333+211j$, $c = -1539+413j$ and $d = -837+439j$.

Let $\mathbf{s}_{-t}$=$\{s_t[(-n)_N]\}$. In (29), when $\mathbf{s}_t$ is replaced by $\mathbf{s}_{-t}$, it constructs a new sequence $\mathbf{s}_{-}$=$\mathbf{s}_{-t} \otimes \mathbf{s}_b \otimes \mathbf{s}_s$, given by

$$\mathbf{s}_{-} = (a,c,d,c,a,d,d,b,b,c,a,b,a).$$

Example 8. *Two construction examples of prime period $N = 2^5 - 1 = 31$ are $\mathbf{t}_1 \otimes \mathbf{t}_3$ and $\mathbf{t}_5 \otimes \mathbf{t}_{15}$, which are*

$$\mathbf{t}_1 \otimes \mathbf{t}_3 = (a,b,b,0,b,c,0,d,b,c,c,c,0,c,d,0,b,0,c,d,c,c,c,0,0,d,c,0,d,0,0), \tag{30}$$

$$\mathbf{t}_5 \otimes \mathbf{t}_{15} = (e, f, f, 0, f, g, 0, h, f, g, g, 0, 0, 0, h, f, f, 0, g, h, g, 0, 0, f, 0, h, 0, f, h, f, f), \quad (31)$$

where $a = -2$, $b = 3$, $c = -1$, $d = 1$, $e = 2$, $f = -1$, $g = 1$ *and* $h = 3$.

7.4. Convolution-Derived PGISs Based on m-Sequences

There exists one-to-one mapping between distinct $m-$sequences and the pattern of degree-2 PGISs. Let us present PGISs of period $N = 2^5 - 1$ as examples for demonstration, of which the six degree-2 PGISs of period $N = 2^5 - 1$ derived from m-sequences are $\{\mathbf{s}_1, \ldots, \mathbf{s}_6\}$, listed in Table 1. Note that the number of different combinations of $\mathbf{s}_l$, $\mathbf{s}_k$ $\in \{\mathbf{s}_1, \ldots, \mathbf{s}_6\}$, $l \neq k$, is 15. We summarize the results of convolution upon two PGISs drawn from the set $\{\mathbf{s}_1, \ldots, \mathbf{s}_6\}$ as follows:

(1) Sequences $\mathbf{s}_1 \otimes \mathbf{s}_2$, $\mathbf{s}_3 \otimes \mathbf{s}_4$ and $\mathbf{s}_5 \otimes \mathbf{s}_6$ are degree-2 PGISs, and the pattern of these three PGISs is the same as that of $\mathbf{s}_{10}$ which is listed in Table 1.
(2) The other 12 kinds of $\mathbf{s}_l \otimes \mathbf{s}_k$ PGISs are degree-3 PGISs, listed in Table 2.
(3) The six sequences $\mathbf{s}_m \otimes \mathbf{s}_m$, $m = 1, \ldots, 6$ are degree-6 PGISs, which are listed in Table 3.
(4) Table 4 presents six patterns of degree-10 PGISs of period 31, where $\mathbf{s}_m \otimes \mathbf{s}_{11}$, $m = 1, \ldots, 6$.

In Section 6.2, the CIDTS-based PGIS construction applies $m-$sequences, $\{\mathbf{m}_1, \ldots, \mathbf{m}_6\}$, directly, for which CCF $R_{b,c}[n]$ is created, and then an adjustment is made by setting $\{\frac{R_{b,c}[n]+1}{4}\}$ to construct PGIS, where $1 \leq b, c \leq 6$. The results are summarized as follows:

(1) Three CCFs, $\{m_1[n]\} \otimes \{m_2^*[(-n)_N]\}$, $\{m_3[n]\} \otimes \{m_4^*[(-n)_N]\}$ and $\{m_5[n]\} \otimes \{m_6^*[(-n)_N]\}$, can be adjusted to construct three degree-5 PGISs, which are $\{\mathbf{t}_{13}, \mathbf{t}_{14}, \mathbf{t}_{15}\}$, presented in (23)–(25). Similarly, three sequences constructed from $\{m_2[n]\} \otimes \{m_1^*[(-n)_N]\}$, $\{m_4[n]\} \otimes \{m_3^*[(-n)_N]\}$ and $\{m_6[n]\} \otimes \{m_5^*[(-n)_N]\}$ are also degree-5 PGISs, denoted by $\{\mathbf{t}_{-13}, \mathbf{t}_{-14}, \mathbf{t}_{-15}\}$.
(2) The 12 distinct CIDTS-based sequences constructed by 12 other kinds of CCFs $\{m_l[n]\} \otimes \{m_k^*[(-n)_N]\}$, $l \neq k$, are all of degree 2, which are denoted by $\{\mathbf{t}_1, \cdots, \mathbf{t}_{12}\}$, listed in Table 1. In addition, 12 kinds of CCFs $\{m_k[n]\} \otimes \{s_l^*[(-n)_N]\}$ will construct other 12 different degree-2 PGISs, which are $\{\mathbf{t}_{-1}, \cdots, \mathbf{t}_{-12}\}$.

7.5. Convolution Derived PGISs Based on CIDTS

In the previous section, the number of CIDTS-based PGISs of period $N = 2^5 - 1$ is 30, which are $\{\mathbf{t}_{13}, \mathbf{t}_{14}, \mathbf{t}_{15}\} \cup \{\mathbf{t}_{-13}, \mathbf{t}_{-14}, \mathbf{t}_{-15}\} \cup \{\mathbf{t}_1, \cdots, \mathbf{t}_{12}\} \cup \{\mathbf{t}_{-1}, \cdots, \mathbf{t}_{-12}\}$. By taking convolution operation $\mathbf{t}_m \otimes \mathbf{t}_k$ upon any two sequences over these 30 PGISs, where the number of different convolution combination of $\mathbf{t}_m$ and $\mathbf{t}_k$ is $\frac{30!}{28! \cdot 2!} = 435$, for $m \neq k$, the number of different degrees and patterns of new generated PGISs can be abundant. The detailed analysis and categorization of these PGISs are not the purpose of this study. For brevity reasons, we present only two results.

(1) The 12 different sequences built from $\mathbf{t}_m \otimes \mathbf{t}_m$, $m = 1, \ldots, 12$, are PGISs of degree 6, listed in Table 3, while three $\mathbf{t}_k \otimes \mathbf{t}_k$, $k = 13, 14, 15$ construct three different PGISs of degree 7 but belong to the same pattern. The pattern of $\mathbf{t}_{13} \otimes \mathbf{t}_{13}$ is listed in Table 3.
(2) When $m \neq k$, some PGISs generated by $\mathbf{t}_m \otimes \mathbf{t}_k$ are provided for comparison, where the degrees of these examples belong to the set $\{1, 2, 4, 5, 6\}$. The degree of PGISs $\mathbf{t}_{13} \otimes \mathbf{t}_{14}$, $\mathbf{t}_{13} \otimes \mathbf{t}_{15}$ and $\mathbf{t}_{15} \otimes \mathbf{t}_{14}$ is 6. The degree of $\mathbf{t}_1 \otimes \mathbf{t}_2$, $\mathbf{t}_1 \otimes \mathbf{t}_4$, $\mathbf{t}_1 \otimes \mathbf{t}_5$, $\mathbf{t}_2 \otimes \mathbf{t}_3$, $\mathbf{t}_2 \otimes \mathbf{t}_4$, $\mathbf{t}_2 \otimes \mathbf{t}_6$, $\mathbf{t}_3 \otimes \mathbf{t}_4$ and $\mathbf{t}_5 \otimes \mathbf{t}_6$ is five. The degree of $\mathbf{t}_3 \otimes \mathbf{t}_5$, $\mathbf{t}_3 \otimes \mathbf{t}_6$, $\mathbf{t}_4 \otimes \mathbf{t}_5$ and $\mathbf{t}_4 \otimes \mathbf{t}_6$ is two. The two PGISs of degree 1 are $\mathbf{t}_1 \otimes \mathbf{t}_6$ and $\mathbf{t}_2 \otimes \mathbf{t}_5$. We do not make a pattern list of these PGISs, for brevity. Finally, two degree-4 examples are $\mathbf{t}_1 \otimes \mathbf{t}_3$ and $\mathbf{t}_5 \otimes \mathbf{t}_{15}$, which are (30) and (31), respectively.

Table 3. The 14 patterns of degree-6 and -7 PGISs of period 31.

PGIS	Sequence Pattern	Coefficients
$\mathbf{s}_1 \otimes \mathbf{s}_1$	$(a,b,b,c,b,d,c,e,b,d,d,c,c,c,e,f,b,c,d,e,d,c,c,f,c,e,c,f,e,f,f)$	$a=-128-26j, b=16+16j,$ $c=-32+2j, d=64+30j,$ $e=112+44j, f=-80-12j$
$\mathbf{s}_2 \otimes \mathbf{s}_2$	$(a,f,f,e,f,c,e,c,f,c,c,d,e,d,c,b,f,e,c,c,c,d,d,b,e,c,d,b,c,b,b)$	
$\mathbf{s}_3 \otimes \mathbf{s}_3$	$(a,e,e,c,e,b,c,d,e,b,b,f,c,f,d,c,e,c,b,d,b,f,f,c,c,d,f,c,d,c,c)$	
$\mathbf{s}_4 \otimes \mathbf{s}_4$	$(a,c,c,d,c,f,d,c,c,f,f,b,d,b,c,e,c,d,f,c,f,b,b,e,d,c,b,e,c,e,e)$	
$\mathbf{s}_5 \otimes \mathbf{s}_5$	$(a,d,d,f,d,e,f,b,d,e,e,c,f,c,b,c,d,f,e,b,e,c,c,c,f,b,c,c,b,c,c)$	
$\mathbf{s}_6 \otimes \mathbf{s}_6$	$(a,c,c,b,c,c,b,f,c,c,c,e,b,e,f,d,c,b,c,f,c,e,e,d,b,f,e,d,f,d,d)$	
$\mathbf{t}_4 \otimes \mathbf{t}_{15}$	$(a,b,b,c,b,d,c,e,b,d,d,f,c,f,e,f,b,c,d,e,d,f,f,f,c,e,f,f,e,f,f)$	$a=-3, b=9, c=2,$ $d=-2, e=8, f=-5$
$\mathbf{t}_1 \otimes \mathbf{t}_1$	$(a,b,b,c,b,d,c,e,b,d,d,0,c,0,e,f,b,c,d,e,d,0,0,f,c,e,0,f,e,f,f)$	$a=11, b=1, c=3,$ $d=2, e=-3, f=-2$
$\mathbf{t}_2 \otimes \mathbf{t}_2$	$(a,0,0,b,0,c,b,d,0,c,c,e,b,e,d,f,0,b,c,d,c,e,e,f,b,d,e,f,d,f,f)$	$a=-9, b=1, c=-1,$ $d=-2, e=5, f=2$
$\mathbf{t}_3 \otimes \mathbf{t}_3$	$(a,0,0,b,0,c,b,d,0,c,c,e,b,e,d,f,0,b,c,d,c,e,e,f,b,d,e,f,d,f,f)$	$a=11, b=1, c=3,$ $d=-2, e=3, f=2$
$\mathbf{t}_4 \otimes \mathbf{t}_4$	$(a,b,b,c,b,d,c,0,b,d,d,e,c,e,0,f,b,c,d,0,d,e,e,f,c,0,e,f,0,f,f)$	$a=-9, b=-1, c=2,$ $d=-2, e=1, f=5$
$\mathbf{t}_5 \otimes \mathbf{t}_5$	$(a,b,b,c,b,d,c,e,b,d,d,f,c,f,e,0,b,c,d,e,d,f,f,0,c,e,f,0,e,0,0)$	$a=-9, b=2, c=-2,$ $d=5, e=1, f=-1$
$\mathbf{t}_6 \otimes \mathbf{t}_6$	$(a,b,b,c,b,0,c,d,b,0,0,e,c,e,d,f,b,c,0,d,0,e,e,f,c,d,e,f,d,f,f)$	$a=11, b=-2, c=-3,$ $d=3, e=2, f=1$
$\mathbf{t}_7 \otimes \mathbf{t}_7$	$(a,b,b,0,b,c,0,d,b,c,c,e,0,e,d,f,b,0,c,d,c,e,e,f,0,d,e,f,d,f,f)$	$a=-9, b=5, c=1,$ $d=2, e=-2, f=-1$
$\mathbf{t}_8 \otimes \mathbf{t}_8$	$(a,b,b,c,b,d,c,e,b,d,d,f,c,f,e,0,b,c,d,e,d,f,f,0,c,e,f,0,e,0,0)$	$a=11, b=2, c=-2,$ $d=-3, e=1, f=3$
$\mathbf{t}_9 \otimes \mathbf{t}_9$	$(a,b,b,0,b,c,0,d,b,c,c,e,0,e,d,f,b,0,c,d,c,e,e,f,0,d,e,f,d,f,f)$	$a=11, b=-3, c=1,$ $d=2, e=-2, f=3$
$\mathbf{t}_{10} \otimes \mathbf{t}_{10}$	$(a,b,b,c,b,0,c,d,b,0,0,e,c,e,d,f,b,c,0,d,0,e,e,f,c,d,e,f,d,f,f)$	$a=-9, b=-2, c=5,$ $d=-1, e=2, f=1$
$\mathbf{t}_{11} \otimes \mathbf{t}_{11}$	$(a,b,b,c,b,d,c,e,b,d,d,0,c,0,e,f,b,c,d,e,d,0,0,f,c,e,0,f,e,f,f)$	$a=-9, b=1, c=-1,$ $d=2, e=5, f=-2$
$\mathbf{t}_{12} \otimes \mathbf{t}_{12}$	$(a,b,b,c,b,d,c,0,b,d,d,e,c,e,0,f,b,c,d,0,d,e,e,f,c,0,e,f,0,f,f)$	$a=11, b=3, c=2,$ $d=-2, e=1, f=-3$
$\mathbf{t}_{13} \otimes \mathbf{t}_{13}$	$(a,b,b,c,b,d,c,e,b,d,d,f,c,f,e,g,b,c,d,e,d,f,f,g,c,e,f,g,e,g,g)$ (degree-7)	$a=-21, b=8, c=-3, d=-17$ $e=2, f=13, g=14$
$\mathbf{s}_{11} \otimes \mathbf{t}_{13}$	$(a,b,b,c,b,d,e,f,b,d,d,g,e,g,f,g,b,e,d,f,d,g,g,g,e,f,g,g,f,g,g)$ (degree-7)	$a=-11, b=4, c=-2, d=-9$ $e=-1, f=1, g=7$

Note that the following pairs of this table have the same sequence pattern: ($\mathbf{s}_5 \otimes \mathbf{s}_5$, $\mathbf{t}_4 \otimes \mathbf{t}_{15}$), ($\mathbf{t}_1 \otimes \mathbf{t}_1$,$\mathbf{t}_{11} \otimes \mathbf{t}_{11}$), ($\mathbf{t}_2 \otimes \mathbf{t}_2$,$\mathbf{t}_3 \otimes \mathbf{t}_3$), ($\mathbf{t}_4 \otimes \mathbf{t}_4$,$\mathbf{t}_{12} \otimes \mathbf{t}_{12}$), ($\mathbf{t}_5 \otimes \mathbf{t}_5$,$\mathbf{t}_8 \otimes \mathbf{t}_8$), ($\mathbf{t}_6 \otimes \mathbf{t}_6$,$\mathbf{t}_{10} \otimes \mathbf{t}_{10}$), ($\mathbf{t}_7 \otimes \mathbf{t}_7$,$\mathbf{t}_9 \otimes \mathbf{t}_9$).

Table 4. Six patterns of degree-10 PGISs of period 31.

PGIS	Sequence Pattern	Coefficients
$\mathbf{s}_1 \otimes \mathbf{s}_{11}$	$(a,b,b,c,d,e,e,f,e,g,f,e,a,h,f,d,f,g,b,i,e,d,d,k,h,d,h,b,f,b,g)$	$a=-6+22j, b=-2-6j,$
$\mathbf{s}_2 \otimes \mathbf{s}_{11}$	$(b,f,b,c,d,b,e,e,a,h,e,g,e,h,f,f,a,g,g,i,e,b,d,d,g,f,d,e,a,l,f)$	$c=2-34j, d=1-27j,$
$\mathbf{s}_3 \otimes \mathbf{s}_{11}$	$(b,b,e,f,e,b,b,d,d,g,c,f,d,f,f,d,b,k,g,e,e,g,h,b,a,e,a,d,i,h,f)$	$e=-4+8j, f=-5+15j,$
$\mathbf{s}_4 \otimes \mathbf{s}_{11}$	$(f,d,e,f,a,b,e,b,f,h,c,a,d,a,e,d,l,d,g,g,b,f,g,g,e,e,b,f,i,h,e)$	$g=-3+j, h=-1-13j,$
$\mathbf{s}_5 \otimes \mathbf{s}_{11}$	$(h,a,e,f,d,e,d,g,d,g,f,i,h,b,c,e,h,b,k,f,b,f,a,g,d,b,b,e,e,f,d)$	$i=-20j, k=-8+36j,$
$\mathbf{s}_6 \otimes \mathbf{s}_{11}$	$(d,b,b,e,f,e,f,f,d,h,f,i,g,l,c,a,h,g,d,a,b,e,e,g,d,e,f,e,g,a,b)$	$l=3-41j$

Table 4 presents six patterns of degree-10 PGISs of period 31.

7.6. Convolution between Different Types of PGISs

This study addresses different construction of PGISs. Therefore, their exist various many different convolution operation applied across different type PGISs. This Section presents only some examples for the purpose of demonstration the versatile of convolution-derived PGISs.

7.6.1. Convolution between Ternary Sequence and CIDTS Derived PGISs

Table 5 presents seven kinds of PGISs obtained from convolution between the perfect ternary sequence and CIDTS-derived PGISs, which are $\mathbf{s}_{11} \otimes \mathbf{t}_{15}$, $\mathbf{s}_{11} \otimes \mathbf{t}_{14}$, $\mathbf{s}_{-11} \otimes \mathbf{t}_{14}$, $\mathbf{s}_{11} \otimes \mathbf{t}_{1}$, $\mathbf{s}_{-11} \otimes \mathbf{t}_{1}$, $\mathbf{s}_{11} \otimes \mathbf{t}_{5}$ and $\mathbf{s}_{-11} \otimes \mathbf{t}_{5}$ for comparison. The patterns are all different, and the degrees of these PGISs are 20, 20, 20, 14, 12, 12 and 12, respectively.

Table 5. Period 31 PGISs of various degrees.

PGIS	Sequence Pattern	Coefficients
$\mathbf{s}_1 \otimes \mathbf{s}_1 \otimes \mathbf{s}_{11}$	$(a,b,c,c,d,e,f,g,h,i,i,k,l,m,n,d,m,p,q,e,b,g,r,s,t,t,u,v,r,w,n)$ (degree-21)	$a=-8+26j, b=-56+12j,$ $c=-104-2j, d=256+103j,$ $e=-152-16j, f=184+82j,$ $g=16+33j, h=424+152j,$ $i=-80+5j, k=40+40j,$ $l=88+54j, m=64+47j,$ $n=160+75j, p=-224-37j,$ $q=-248-44j, r=-32+19j,$ $s=-344-72j, t=208+89j,$ $u=-128-9j, v=136+68j,$ $w=-200-30j$
$\mathbf{s}_2 \otimes \mathbf{s}_2 \otimes \mathbf{s}_{11}$	$(c,g,a,b,u,a,w,e,a,d,a,m,v,d,t,f,c,n,h,e,k,q,r,d,n,l,m,p,b,h,i)$ (degree-20)	$a=-8+26j, b=-56+12j,$ $c=232+96j, d=256+103j,$ $e=-152-16j, f=-272-51j,$ $g=16+33j, h=112+61j,$ $i=-80+5j, k=40+40j,$ $l=-320-65j, m=64+47j,$ $n=160+75j, p=-296-58j,$ $q=-248-44j, r=-32+19j,$ $w=-200-30j, t=208+89j,$ $u=-128-9j, v=136+68j$
$\mathbf{s}_{11} \otimes \mathbf{t}_{15}$	$(a,b,c,d,e,f,g,h,e,d,h,c,g,i,j,k,l,m,n,p,j,q,r,s,p,t,f,u,0,l,q)$ (degree-20)	$a=7, b=2, c=8, d=5,$ $e=-9, f=4, g=-7, h=-5,$ $i=10, j=6, k=-16, l=-1,$ $m=12, n=16, p=3, q=1,$ $r=-2, s=11, t=-4, u=-6$
$\mathbf{s}_{11} \otimes \mathbf{t}_{14}$	$(a,b,a,c,d,e,f,g,h,i,e,j,j,k,i,h,l,m,n,0,p,c,q,r,s,t,u,q,p,k,g)$ (degree-20)	$a=4, b=7, c=-5, d=-16,$ $e=6, f=-4, g=1, h=-9,$ $i=5, j=3, k=-1, l=10,$ $m=16, n=11, p=8, q=-7,$ $r=12, s=-2, t=-6, u=2$
$\mathbf{s}_{-11} \otimes \mathbf{t}_{14}$	$(a,b,c,b,d,e,f,g,h,i,j,k,l,i,c,h,m,k,e,n,p,p,c,q,r,s,t,t,u,0,l)$ (degree-20)	$a=1, b=8, c=-9, d=15,$ $e=-8, f=10, g=-4, h=-3,$ $i=-5, j=5, k=2, l=-1,$ $m=7, n=3, p=4, q=13,$ $r=12, s=14, t=2, u=-7$
$\mathbf{s}_{11} \otimes \mathbf{t}_{1}$	$(a,b,0,b,c,d,0,e,f,f,e,g,h,i,j,k,l,l,b,m,c,m,m,b,j,0,c,n,d,m,g)$ (degree-14)	$a=3, b=-1, c=2, d=-2,$ $e=-4, f=3, g=4, h=1,$ $i=6, j=8, k=-3, l=5,$ $m=-3, n=-5$
$\mathbf{s}_{-11} \otimes \mathbf{t}_{1}$	$(a,b,c,d,e,d,0,a,0,f,f,0,g,f,g,0,c,h,h,e,i,g,e,j,f,k,d,l,f,0,c)$ (degree-12)	$a=6, b=3, c=-3, d=-2,$ $e=1, f=-1, g=-4, h=7,$ $i=5, j=9, k=2, l=4$
$\mathbf{s}_{11} \otimes \mathbf{t}_{5}$	$(a,b,a,a,c,d,e,f,g,h,d,i,d,j,h,i,a,k,j,d,0,a,h,l,b,i,c,0,0,c,k)$ (degree-12)	$a=2, b=5, c=-3, d=-2,$ $e=-6, f=-4, g=4, h=3,$ $i=1, j=9, k=-1, l=-5$

Table 5. *Cont.*

PGIS	Sequence Pattern	Coefficients
$\mathbf{s}_{-11} \otimes \mathbf{t}_5$	$(a, b, c, d, e, f, g, b, c, g, d, 0, 0, h, i, i, g, b, g, c, a, g, i, a, i, j, 0, k, 0, e, l)$ (degree-12)	$a = 2, b = 3, c = 4, d = -3,$ $e = -2, f = 5, g = -1, h = -10,$ $i = 1, j = 6, k = 9, l = -6$
$\mathbf{s}_{cy} \otimes \mathbf{s}_{11}$	$(a, a, b, c, d, b, e, f, d, g, c, h, i, k, c, d, k, m, m, h, b, f, i, m, i, e, a, e, h, k, f)$ (degree-11)	$a = -13 - 13j, b = -14 - 14j,$ $c = -6 - 6j, d = 30 + 30j,$ $e = 21 + 21j, f = 7 + 7j,$ $g = -19 - 19j, h = -7 - 7j,$ $i = 2 + 2j, k = -12 - 12j,$ $m = -35 - 35j$

Table 5 presents various higher different-degree PGISs obtained from convolution operation.

7.6.2. Convolution between Ternary Sequence and *m*-Sequences Derived PGISs

Table 5 presents two kinds of PGISs obtained from convolution between the perfect ternary sequence and *m*-sequence-derived PGISs, which are $\mathbf{s}_1 \otimes \mathbf{s}_1 \otimes \mathbf{s}_{11}$ and $\mathbf{s}_2 \otimes \mathbf{s}_2 \otimes \mathbf{s}_{11}$. The degrees are 21 and 20, respectively.

7.6.3. Convolution between Ternary Sequence and Cyclotomic Class PGIS

Table 5 presents also one PGIS obtained from convolution between the perfect ternary sequence and degree-3 PGIS using the cyclotomic class of order two, which is $\mathbf{s}_{cy} \otimes \mathbf{s}_{11}$ and the degree is 11.

7.6.4. Convolution between CIDTS Derived and Cyclotomic Class PGIS

The 15 different PGISs obtained from convolution between CIDTS-derived PGISs, which are $\{\mathbf{t}_1, \cdots, \mathbf{t}_{15}\}$, and degree-3 PGIS using cyclotomic class of order two $\mathbf{s}_{cy}$ can be distributed into two degree-7 and degree-6 groups, of which six PGISs that belong to set $\{\mathbf{s}_{cy} \otimes \mathbf{t}_m, m = 2, 4, 5, 7, 10, 11\}$ are of degree 6 and the rest of the other nine PGISs are of degree 7. The patterns of these PGISs belong to those patterns listed in Table 3.

8. Conclusions

A review study of prime period PGIS construction is addressed in this paper. Prime period sequences can serve as the fundamental tool to construct arbitrary composite period sequences. We introduce the novel idea of two different systematic and nonsystematic approaches for construction of prime period PGISs. The systematic approach encounters difficulty to solve constraint equations when the degree of sequence is lager than 3; however, the merit of this approach is that both the degree and pattern of a sequence are known, and PGISs of degrees 1, 2, 3 and 5 examples are presented for demonstration. The nonsystematic approach can contribute abundant numbers of degrees and patterns to the constructed PGISs, but both the degree and pattern might vary. We provide PGISs of different patterns and degree-4 and other higher-degree—6, 7, 10, 11, 12, 14, 20 and 21—examples to show the results of nonsystematic approach. From the PGIS application point of view, the proposed systematic and nonsystematic schemes can be combined to construct efficiently abundant PGISs with various degrees and patterns for the associated different applications.

Finally, we emphasize that one can construct abundant PGISs of different degrees and patterns by convolution between two PGISs, and how can we govern the nonsystematic approach to control the variation of degrees and patterns should be our future work.

Author Contributions: supervision,review and editing, H.-H.C.; project administration, S.G.; methodology, M.Z.; validation, P.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Chu, D. Polyphase codes with good periodic correlation properties. *IEEE Trans. Inf. Theory* **1972**, *18*, 531–532.
2. Frank, R.L.; Zadoff, S.A. Phase shift pulse codes with good periodic correlation properties. *IEEE Trans. Inf. Theory* **1962**, *8*, 381–382.
3. Gong, G.; Huo, F.; Yang, Y. Large zero autocorrelation zones of Golay sequences and their applications. *IEEE Trans. Commun.* **2013**, *61*, 3967–3979.
4. Yu, N.Y.; Gong, G. New binary sequences with optimal autocorrelation magnitude. *IEEE Trans. Inf. Theory* **2008**, *54*, 4771–4779.
5. Yan, T.; Xiao, G. Divisible difference sets, relative difference sets and sequences with ideal autocorrelation. *Inf. Sci.* **2013**, *249*, 143–147.
6. Fan, P.Z.; Darnell, M. *Sequences Design for Communications Applications*; Wiley: New York, NY, USA, 1996.
7. Fan, P.Z.; Darnell, M. Maximal length sequences over Gaussian integers. *Inst. Electron. Lett.* **2000**, *36*, 552–553.
8. Choi, S.-H.; Baek, J.-S.; Han, J.-S.; Seo, J.-S. Channel estimations using orthogonal codes for AF multiple-relay networks over frequency-selective fading channels. *IEEE Trans. Veh. Technol.* **2014**, *63*, 417–423.
9. Qureshi, S.U.H. Fast start-up equalization with period training sequences. *IEEE Trans. Inf. Theory* **1997**, *23*, 553–563.
10. Milewski, A. Periodic sequences with optimal properties for channel estimation and fast start-up equalization. *IBM J. Res. Develop.* **1983**, *27*, 425–431.
11. Lin, J.-C. Initial Synchronization Assisted by Inherent Diversity over Time-Varying Frequency-Selective Fading Channels. *IEEE Trans. Wirel. Commun.* **2014**, *13*, 2518–2529.
12. Popovic, B.M. Generalized chirp-like polyphase sequences with optimum correlation properties. *IEEE Trans. Inf. Theory* **1992**, *38*, 1406–1409.
13. Kim, H.; Han, Y.; Kim, Y.; Bang, S.C. Sequence hopping cell search scheme for OFDM cellular systems. *IEEE Trans. Wirel. Commun.* **2008**, *7*, 1483–1489.
14. Li, C.-P.; Wang, S.-H.; Wang, C.-L. Novel low-complexity SLM schemes for PAPR reduction in OFDM systems. *IEEE Trans. Signal Process.* **2010**, *58*, 2916–2921.
15. Wang, S.-H.; Lee, K.-C.; Li, C.-P.; Su, H.-J. A novel low-complexity precoded OFDM system with reduced PAPR. *IEEE Trans. Signal. Process.* **2015**, *63*, 1366–1376.
16. Huber, K. Codes over Gaussian integers. *IEEE Trans. Inf. Theory* **1994**, *40*, 207–216.
17. Zeng, F.; Zeng, X.; Zhang, Z.; Xuan, G. 8-QAM+ periodic complementary sequence sets. *IEEE Commun. Lett.* **2012**, *16*, 83–85.
18. Li, Y. A construction of general QAM Golay complementary sequences. *IEEE Trans. Inf. Theory* **2010**, *56*, 5765–5771.
19. Lee, H.; Golomb, S.W. A new construction of 64-QAM Golay complementary sequences. *IEEE Trans. Inf. Theory* **2006**, *52*, 1663–1670.
20. Luke, H.D.; Schotten, H.D.; Hadinejad-Mahram, H. Binary and quadriphase sequences with optimal autocorrelation properties: A survey. *IEEE Trans. Inf. Theory* **2003**, *49*, 3271–3282.
21. Hu, W.-W.; Wang, S.-H.; Li, C.-P. Gaussian integer sequences with ideal periodic autocorrelation functions. *IEEE Trans. Signal Process.* **2012**, *60*, 6074–6079.
22. Yang, Y.; Tang, X.; Zhou, Z. Perfect Gaussian integer sequences of odd prime length. *IEEE Signal Process. Lett.* **2012**, *19*, 615–618.
23. Ma, X.; Wen, Q.; Zhang, J.; Zuo, H. New perfect Gaussian integer sequences of period pq. *IEICE Trans. Fundam.* **2013**, *E96-A*, 2290–2293.
24. Chang, H.-H.; Li, C.-P.; Lee, C.-D.; Wang, S.-H.; Wu, T.-C. Perfect Gaussian integer sequences of arbitrary composite length. *IEEE Trans. Inf. Theory* **2015**, *61*, 4107–4115.
25. Lee, C.-D.; Huang, Y.-P.; Chang, Y.; Chang, H.-H. Perfect Gaussian integer sequences of odd period 2^{m-1}. *IEEE Signal Process. Lett.* **2015**, *12*, 881–885.
26. Lee, C.-D.; Li, C.-P.; Chang, H.-H.; Wang, S.-H. Further results on degree-2 Perfect Gaussian integer sequences. *IET Commun.* **2016**, *10*, 1542–1552.
27. Lee, C.-D.; Hong, S.-H. Generation of long perfect Gaussian integer sequences. *IEEE Signal Process. Lett.* **2017**, *24*, 515–519.
28. Lee, C.-D.; Chen, Y.-H. Families of Gaussian integer sequences with high energy efficiency. *IET Commun.* **2016**, *10*, 416–2421.
29. Pei, S.-C.; Chang, K.-W. Perfect Gaussian Integer Sequences of Arbitrary Length. *IEEE Signal Process. Lett.* **2015**, *22*, 1040–1044.
30. Chang, H.-H.; Chang, K.-J.; Li, C.-P. Construction of Period qp PGISs With Degrees Equal To or Larger Than Four. *IEEE Access* **2018**, *65*, 3723–3733.
31. Chang, H.-H.; Lin, S.-C.; Lee, C.-D. A CDMA scheme based on perfect Gaussian integer sequences. *AEÜ Int. J. Electron. Commun.* **2017**, *75*, 70–81.
32. Hsia, C.-H.; Lou, S.-J.; Chang, H.-H.; Xuan, D. Novel Hybrid Public/Private Key Cryptography Based on Perfect Gaussian Integer Sequences. *IEEE Access* **2021**, *9*, 145045–145059.
33. Xuan, D.; Chang, H.-H.; Huang, G. Novel Zero Circular Convolution Sequences for Detection and Channel Estimations. *IEEE Access* **2023**, *11*, 48276–48291.
34. Howie, J.M. *Fields and Galois Theory*; Springer: London, UK, 2006.
35. Li, C.-P.; Chang, K.-J.; Chang, H.-H.; Chen, Y.-M. Perfect Sequences of Odd Prime Length. *IEEE Signal Process. Lett.* **2018**, *25*, 966–969.
36. Ipatov, V.P. Ternary sequences with ideal autocorrelation properties. *Radio Eng. Electron. Phys.* **1979**, *24*, 75–79.

37. Ipatov, V.P. Contribution to the theory of sequences with perfect period autocorrelation properties. *Radio Eng. Electron. Phys.* **1980**, *24*, 31–34.
38. Schroeder, M.R. *Number Theory in Science and Communications*; Springer: Berlin/Heidelberg, Germany, 1997.
39. Lancaster, P. *Theory of Matrices*; Academic Press: New York, NY, USA, 1969.
40. Davis, P.J. *Circulant Matrices*; Wiley-Interscience: New York, NY, USA, 1979.