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Hilbert Graceful Labeling on Complete Multipartite Graph

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Authors' contributions

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

Let G be a simple, finite, connected, undirected, non-trivial graph with p vertices and q edges. $V(G)$ be the vertex set and $E(G)$ be the edge set of G. The n^{th} Hilbert number is denoted by H_n and is defined by $H_n = 4(n-1)+1$ where $n \geq 1$. A Hilbert graceful labeling is an injective function $\mathcal{H}$ from the vertex set $V(G)$ to a set of Hilbert number $\{x : x = 4(i-1)+1, 1 \leq i \leq 2q\}$ which induces a bijective function $\mathcal{H}^*$ from the set $E(G)$ to the set of number $\{1, 2, 3, 4, \dots, q\}$, where for each edge $uv \in E(G)$ with $u, v \in V(G)$ applies $\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$. A graph with Hilbert graceful labeling is called a Hilbert graceful graph. This research aims to prove that some complete multipartite graphs are Hilbert graceful by providing systematic proofs and clear constructions of the labeling functions. Our contributions include the identification and characterization of these graphs, expanding the class of graphs known to exhibit Hilbert graceful properties, and providing illustrative examples to support our findings.

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Keywords: Hilbert numbers; graceful labelling; hilbert graceful labelling; hilbert graceful graph.

1 Introduction

A graph G consists of a finite set of vertices $V(G)$ and a set of edges $E(G)$ consisting of distinct, unordered pairs of vertices" [1]. The graph discussed in this paper is a simple, undirected, and finite graph. $|V(G)|$ represents the number of vertices on graph G, and the number of edges on graph G is represented by $|E(G)|$. Graph labeling has been studied since the 60s. Graph labeling is a branch of graph theory that has continued to develop. Labeling on a graph is the assignment of an integer value to the elements of the graph, usually a positive integer. Alex Rosa first discovered graceful labeling in 1967 [2]. Since this discovery, many researchers have been interested in looking for graceful labeling constructs and their variations. Several graphs with graceful labeling include tree graphs with vertices less or equal to 35, circle graph C_n for $n = 0(mod\ 4)$ or $n = 3(mod\ 4)$, and wheel graph W_n. Another class of graphs known to have graceful labeling can be seen in the survey conducted by Gallian [3]. The following shows some relevant research: graceful labeling of paths, graceful labeling of pendant graphs, vertex graceful labeling of caterpillar graphs, graceful labeling on torch graph, counting graceful labelings of trees, and other results on super graceful labeling of graphs [4-11]. Motivated by the above articles, this paper introduces a new type of graceful labeling called Hilbert graceful labeling and studies Hilbert graceful labeling for some complete bipartite graphs" [12,13].

2 Definition

Definition 2.1: The graph $K_{1,m,n}$ is obtained by connecting all the vertices of complete bipartite graph $K_{m,n}$ to a vertex v.

Definition 2.2: The graph $K_{2\ ,m,n}$ is obtained by connecting all the vertices of complete bipartite graph $K_{m,n}$ to a vertices u and v.

Definition 2.3: The graph $K_{1,1,m,n}$ is obtained by connecting all the vertices of complete bipartite graph $K_{m,n}$ to the end points of an edge uv.

Definition 2.4: A graph G with the vertex set $V(G) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{w_k : 1 \le k \le l\}$ and the edge set $E(G) = \{u_i v_j : 1 \le i \le m, 1 \le j \le n\} \cup \{v_j\, w_k : 1 \le j \le n, 1 \le k \le l\}$ such that $|V(G)| = m + n + l$ and $|E(G)| = m(n + l)$ is denoted by $K_{m,n,l}$.

3 Main Result

Theorem 3.1: The graph $K_{1,m,n}$ admits Hilbert graceful labeling.

Proof: Let G be a $K_{1,m,n}$ graph.

$$V(G) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{v\} \text{ and}$$
$$E(G) = \{u_i\, v_j : 1 \le i \le m;\ 1 \le j \le n\} \cup \{v\, u_i : 1 \le i \le m\} \cup \{v\, v_j : 1 \le j \le n\}.$$
$$|V(G)| = m + n + 1 \text{ and } |E(G)| = m\,n + m + n$$

We define a function $\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$

The vertex labeling is as follows:

$$\mathcal{H}(u_i) = 4[i] + 1 \qquad 1 \le i \le m$$
$$\mathcal{H}(v_j) = 4[m + (m+1)\, j] + 1 \qquad 1 \le j \le n$$
$$\mathcal{H}(v) = 1 \qquad 1 \le k \le l$$

By above labeling pattern, we observed that function

$\mathcal{H} : V(G) \rightarrow \{x : x = 4(i-1) + 1, 1 \leq i \leq 2q\}$ is $1 - 1$.

From the induced function $\mathcal{H}^*: E(G) \rightarrow \{1, 2, 3, 4, \dots, q\}$, we get the edge labels as follows.

Table 1. Edge labels of the graph $K_{1,m,n}$

$\mathcal{H}^*$	Edge Labels	Value of i, j and k
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	$m - i + (m+1)j$	$1 \leq i \leq m, 1 \leq j \leq n$
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v)\rvert$	i	$1 \leq i \leq m$
$\lvert\mathcal{H}(v_j) - \mathcal{H}(v)\rvert$	$m + (m+1)j$	$1 \leq j \leq n$

From the above Table 1, we observe that $\mathcal{H}^*: E(G) \rightarrow \{1, 2, 3, 4, \dots, q\}$ defined by $\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$ is a bijective. Hence, $\mathcal{H}$ is Hilbert graceful labeling and the graph $K_{1,m,n}$ is Hilbert graceful graph.

Example 3.1: Hilbert graceful labeling of the graph $K_{1,3,4}$

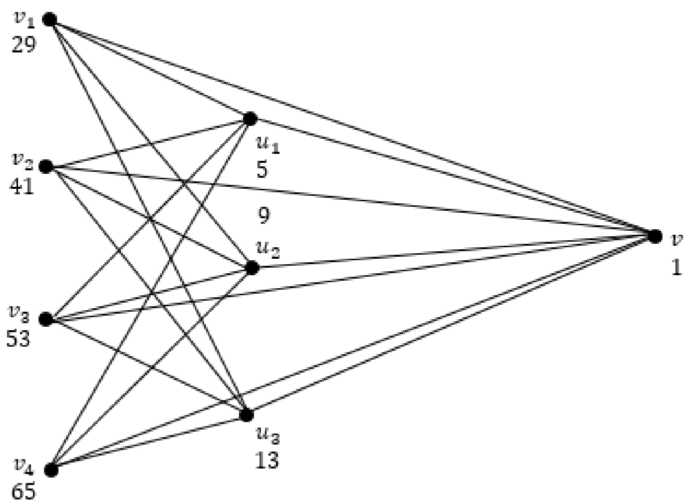

Fig. 1. The graph $K_{1,3,4}$

Theorem 3.2: The graph $K_{2,m,n}$ admits Hilbert graceful labeling.

Proof: Let G be a $K_{2,m,n}$ graph.

$V(G) = \{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\} \cup \{u, v\}$ and
$E(G) = \{u_i\, v_j: 1 \leq i \leq m;\ 1 \leq j \leq n\} \cup \{u\, u_i: 1 \leq i \leq m\} \cup \{u\, v_j: 1 \leq j \leq n\}$
$\cup \{v\, u_i: 1 \leq i \leq m\} \cup \{v\, v_j: 1 \leq j \leq n\}$.
$|V(G)| = m + n + 2$ and $|E(G)| = m\,n + 2m + 2n$

We define a function $\mathcal{H} : V(G) \rightarrow \{x : x = 4(i-1) + 1, 1 \leq i \leq 2q\}$

The vertex labeling is as follows:

$\mathcal{H}(u_i) = 4i + 1 \qquad 1 \leq i \leq m$
$\mathcal{H}(v_1) = 1$
$\mathcal{H}(v_j) = 4[(m+2)j - 2] + 1 \qquad 2 \leq j \leq n$
$\mathcal{H}(u) = 4[mn + 2m + 2n] + 1$

$$\mathcal{H}(v) = 4[mn + m + 2n - 1] + 1$$

By above labeling pattern, we observed that function

$$\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \leq i \leq 2q\} \text{ is } 1-1.$$

From the induced function $\mathcal{H}^*: E(G) \to \{1, 2, 3, 4, \dots, q\}$, we get the edge labels as follows.

Table 2. Edge labels of the graph $K_{2,m,n}$

$\mathcal{H}^*$	Edge Labels	Value of i, j and k
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	i	$1 \leq i \leq m, j = 1$
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	$(m+2)j - i - 2$	$2 \leq j \leq n;\ 1 \leq i \leq m$
$\lvert\mathcal{H}(u_i) - \mathcal{H}(u)\rvert$	$mn + 2m + 2n - i$	$1 \leq i \leq m$
$\lvert\mathcal{H}(v_j) - \mathcal{H}(u)\rvert$	m	$j = 1$
$\lvert\mathcal{H}(v_j) - \mathcal{H}(u)\rvert$	$mn + 2m + 2n + 2 - (m+2)j$	$2 \leq j \leq n$
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v)\rvert$	$mn + m + 2n - 1 - i$	$1 \leq i \leq m$
$\lvert\mathcal{H}(v_j) - \mathcal{H}(v)\rvert$	$mn + m + 2n - 1$	$j = 1$
$\lvert\mathcal{H}(v_j) - \mathcal{H}(v)\rvert$	$mn + m + 2n + 1 - (m+2)j$	$2 \leq j \leq n$

From the above Table 2, we observe that $\mathcal{H}^*: E(G) \to \{1, 2, 3, 4, \dots, q\}$ defined by $\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$ is a bijective. Hence, $\mathcal{H}$ is Hilbert graceful labeling and the graph $K_{2,m,n}$ is Hilbert graceful graph.

Example 3.2: Hilbert graceful labeling of the graph $K_{2,m,n}$

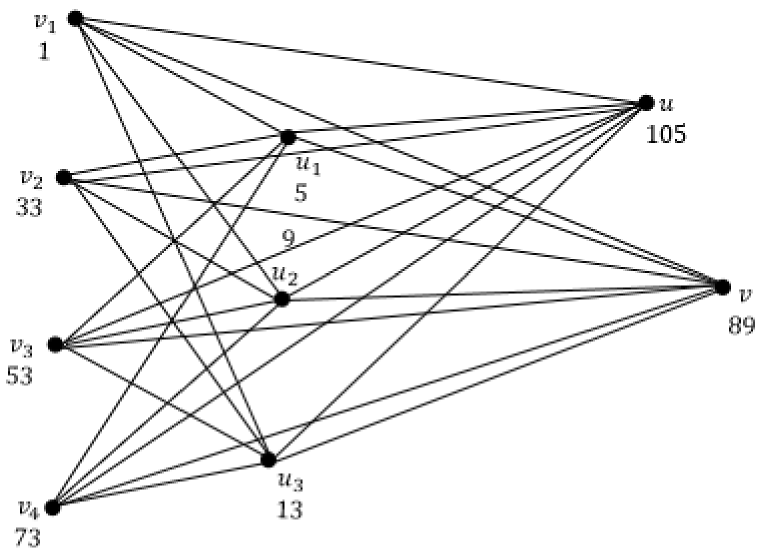

Fig. 2. The graph $K_{2,m,n}$

Theorem 3.3: The graph $K_{1,1,m,n}$ admits Hilbert graceful labeling.

Proof: Let G be a $K_{1,1,m,n}$ graph.

$$V(G) = \{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\} \cup \{u, v\} \text{ and}$$

$E(G) = \{u_i\, v_j : 1 \le i \le m;\ 1 \le j \le n\} \cup \{u\, u_i : 1 \le i \le m\} \cup \{u\, v_j : 1 \le j \le n\}$
$\cup \{v\, u_i : 1 \le i \le m\} \cup \{v\, v_j : 1 \le j \le n\} \cup \{u\, v\}.$
$|V(G)| = m + n + 2$
$|E(G)| = m\, n + 2m + 2n + 1$

We define a function $\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$

The vertex labeling is as follows:

$\mathcal{H}(u_i) = 4i\, d + 1 \qquad 1 \le i \le m$
$\mathcal{H}(v_j) = 4[m + (m+2)j] + 1 \qquad 1 \le j \le n$
$\mathcal{H}(u) = 4[mn + 2m + 2n + 1] + 1$
$\mathcal{H}(v) = 1$

By above labeling pattern, we observed that function

$\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$ is $1 - 1$.

From the induced function $\mathcal{H}^*: E(G) \to \{1, 2, 3, 4, \ldots, q\}$, we get the edge labels as follows.

Table 3. Edge labels of the graph $K_{1,1,m,n}$.

$\mathcal{H}^*$	Edge Labels	Value of i, j and k
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	$m - i + (m+2)j$	$1 \le i \le m, 1 \le j \le n$
$\lvert\mathcal{H}(u_i) - \mathcal{H}(u)\rvert$	$mn + 2m + 2n + 1 - i$	$1 \le i \le m$
$\lvert\mathcal{H}(v_j) - \mathcal{H}(u)\rvert$	$mn + m + 2n + 1 - (m+2)j$	$1 \le j \le n$
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v)\rvert$	i	$1 \le i \le m$
$\lvert\mathcal{H}(v_j) - \mathcal{H}(v)\rvert$	$m + (m+2)j$	$1 \le j \le n$
$\lvert\mathcal{H}(u) - \mathcal{H}(v)\rvert$	$mn + 2m + 2n + 1$	

From the above Table 3, we observe that $\mathcal{H}^*: E(G) \to \{1, 2, 3, 4, \ldots, q\}$ defined by
$\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$ is a bijective. Hence, $\mathcal{H}$ is Hilbert graceful labeling and the graph $K_{1,1,m,n}$ is Hilbert graceful graph.

Example 3.3: Hilbert graceful labeling of the graph $K_{1,1,3,4}$.

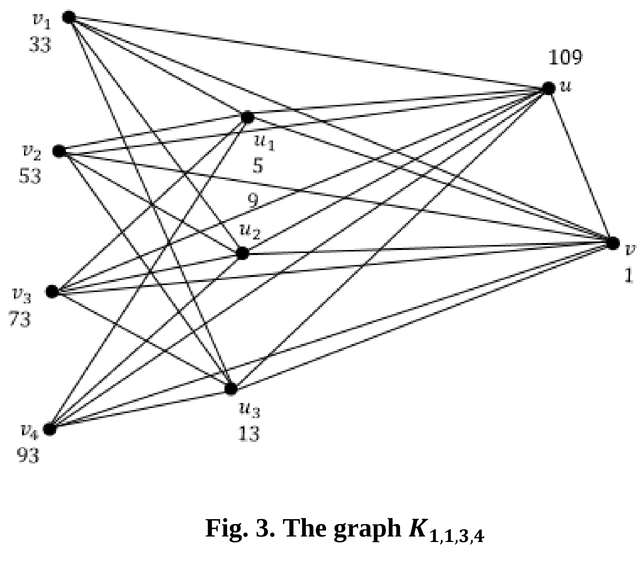

Fig. 3. The graph $K_{1,1,3,4}$

Theorem 3.4: The graph $K_{m,n,l}$ admits Hilbert graceful labeling.

Proof: Let G be a $K_{m,n,l}$ graph.

$V(G) = \{u_i : 1 \le i \le m\} \cup \{v_j : 1 \le j \le n\} \cup \{w_k : 1 \le k \le l\}$ and
$E(G) = \{u_i\, v_j : 1 \le i \le m, 1 \le j \le n\} \cup \{u_i\, w_k : 1 \le i \le m, 1 \le k \le l\}$.
$|V(G)| = m + n + l$
$|E(G)| = m(n + l)$

We define a function $\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$

The vertex labeling is as follows:

$\mathcal{H}(u_i) = 4\,[i-1] + 1 \qquad 1 \le i \le m$
$\mathcal{H}(v_j) = 4[mj] + 1 \qquad 1 \le j \le n$
$\mathcal{H}(w_k) = 4[mn + mk] + 1 \qquad 1 \le k \le l$

By above labeling pattern, we observed that function

$\mathcal{H} : V(G) \to \{x : x = 4(i-1) + 1, 1 \le i \le 2q\}$ is $1 - 1$.

From the induced function $\mathcal{H}^* : E(G) \to \{1, 2, 3, 4, \ldots, q\}$, we get the edge labels as follows.

Table 4. Edge labels of the graph $K_{m,n,l}$

$\mathcal{H}^*$	Edge Labels	Value of i, j and k
$\lvert\mathcal{H}(u_i) - \mathcal{H}(v_j)\rvert$	$1 - i + mj$	$1 \le i \le m, 1 \le j \le n$
$\lvert\mathcal{H}(u_i) - \mathcal{H}(w_k)\rvert$	$mn + 1 - i + mk$	$1 \le i \le m, 1 \le k \le l$

From the above Table 4, we observe that $\mathcal{H}^* : E(G) \to \{1, 2, 3, 4, \ldots, q\}$ defined by
$\mathcal{H}^*(uv) = \frac{1}{4}|\mathcal{H}(u) - \mathcal{H}(v)|$ is a bijective. Hence, $\mathcal{H}$ is Hilbert graceful labeling and the graph $K_{m,n,l}$ is Hilbert graceful graph.

Example 3.4: Hilbert graceful labeling of the graph $K_{3,4,4}$.

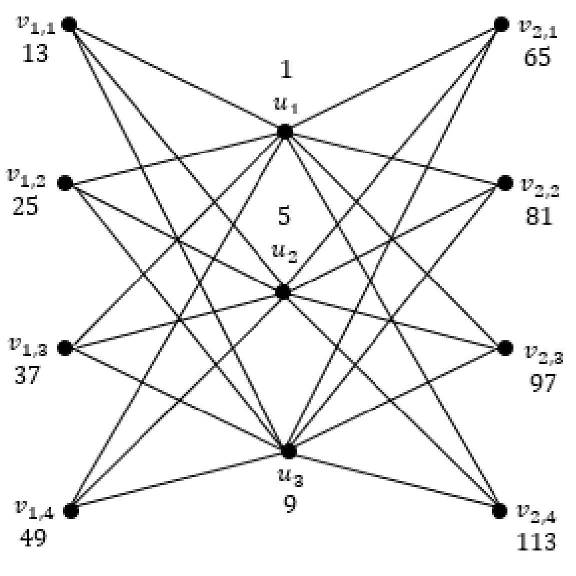

Fig. 4. The graph $K_{3,4,4}$

4 Conclusion

In this paper, we proved that certain complete multipartite graphs are Hilbert graceful graphs. The proofs were presented systematically, with clear definitions and step-by-step constructions of the labeling functions, ensuring a comprehensive understanding of the results. We utilized illustrations to demonstrate the labeling patterns, which provided visual clarity and reinforced the validity of our findings. This work contributes to the field of graph theory by expanding the class of graphs known to possess the Hilbert graceful property. As part of our future research, we aim to explore Hilbert graceful labeling in other families of graphs, potentially uncovering new insights and broadening the scope of this concept.

Disclaimer (Artificial Intelligence)

I confirm that no generative AI technologies, such as Large Language Models (e.g., ChatGPT, Copilot) or text-to-image generators, have been used in the writing or editing of this manuscript.

Competing Interests

Author has declared that no competing interests exist.

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