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The Improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -Expansion Method for The Non-dissipative Double Dispersive Equation in Microstructured Solids

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Authors' contributions

This work was carried out in collaboration between both authors. Author LY designed the study, performed the statistical analysis, wrote the protocol, wrote the manuscript, and managed literature searches. Author SL was responsible for supervising the planning and implementation of research activities. Both authors read and approved the final manuscript.

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Abstract

In microstructured solids, the non-dissipative double dispersive equation is a fourth-order non-linear partial differential equation that arises in the study of non-dissipative strain wave propagation. Seeking the exact solution of a nonlinear partial differential equations with a physical background is helpful to understand the motion law of matter and to explain the corresponding physical phenomena scientifically. This equation has been studied widely, and there have been a lot of research methods to find exact solutions to this equation. In this manuscript, we try to study the non-dissipative double dispersive equation by the improved

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$\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method. As far as our known, no one has used this method to study the non-dissipative double dispersive equation, partly because of the complicated calculation. We obtain a lot of exact traveling wave solutions, including hyperbolic function solutions, trigonometric function solutions, exponential function solutions, and rational function solutions. Compared with the other methods, some solutions obtained in this manuscript are consistent with existing solutions, which shows the effectiveness of the improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method. Through this method, Our manuscript obtains more general and new exact solutions. These solutions may play an important role in engineering and physics. What's more, we plot the 3D graphs of some of the solutions obtained in this manuscript, which helps us understand the physical phenomenon of the non-dissipative double dispersive equation. In the future, we will continue to explore its physical significance from the new analytical solution we have obtained.

Keywords: *The non-dissipative double dispersive equation; the improved* $\tan\left(\frac{\Phi(\xi)}{2}\right)$ *-expansion method; exact solutions; microstructured solids.*

1 Introduction

In this manuscript, we study the non-dissipative double dispersive equation in the microstructured solid as[1]

$$u_{tt} - u_{xx} - \varepsilon\alpha_1(u^2)_{xx} + \varepsilon\alpha_3 u_{xxxx} - \varepsilon\alpha_4 u_{xxtt} = 0, \tag{1.1}$$

where $u = u(x,t)$ is a function of two variables x, t, and ε represent the elastic strains and α_j are constants. Microstructure materials such as alloys, ceramics, grains, and functionally gradient materials have been used widely. The strain wave equation is a fourth-order non-linear partial differential equation that arises in the study of non-dissipative strain wave propagation in microstructured solids. The nonlinear strain wave equation in microstructured solids which is governed as[1]

$$u_{tt} - u_{xx} - \varepsilon\alpha_1(u^2)_{xx} - \gamma\alpha_2 u_{xxt} + \delta\alpha_3 u_{xxxx} - (\delta\alpha_4 - \gamma^2\alpha_7)u_{xxtt} + \gamma\delta(\alpha_5 u_{xxxxt} + \alpha_6 u_{xxttt}) = 0. \tag{1.2}$$

where ε represent the elastic strains, δ gives the ratio between microstructure size and the wavelength, γ is the coefficient of dissipation, and α_j for $j = 1, 2, \cdots 7$ are constants. The balance between non-linearity and dispersion occurs when $\delta = O(\varepsilon)$. If the condition $\gamma = 0$ is added, we have the non-dissipative case governed by the double dispersive equation (1.1).

Since the non-dissipative double dispersive equation has been used widely, it is meaningful to study its exact solutions. Over the past few years, there were many effective methods to obtain various types of the exact solutions of Eq. (1.1), see[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 14, 16, 17, 18, 19, 20]. Hafez et al. [4] obtained the solitary wave solutions of topological kink, singular kink, nontopological bell type solutions, solitons, compacton, periodic solutions, and solitary wave solutions of rational functions by the exponential expansion method. In[14], Seadawy et al. applied the improved form of the simple equation and modified F-expansion techniques for obtaining exact solutions of the non-dissipative double dispersive equation. Hameedullah et al. [20] applied the Sardar sub-equation technique to derive solutions of singular solitons, bright solitons, multi-M-shaped solitons, and the interaction of singular solitons with periodic solitons and bright periodic solitons. Solutions of nonlinear partial differential equations play an important role in modeling scientific and engineering problems. In the past decades, various techniques for solving nonlinear differential equations have been proposed, such as the homogeneous balance method[21], the sine-cosine method[22], the Jacobi elliptic function method[23], the (G'/G)-expansion method [24]. In [25], Manafian et al. proposed a new expansion method named the improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method for solving the nonlinear partial differential equation. At present, the improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method has been used to seek exact solutions of well-known nonlinear partial differential equations[26, 27, 28, 29, 30, 31, 32, 33].

As far as our known, no scholars have studied the non-dissipative double dispersive equation in microstructured solids by the improved tan $\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method. The objective of this manuscript is to apply the improved tan $\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method to obtain the exact solutions to the non-dissipative double dispersive equation. Through this method, we got a lot of traveling wave solutions, some of which are new solutions. By supposing that an ordinary differential equation obtained by transformation has a formal solution

$$u(\xi) = S(\Phi) = \sum_{k=0}^{N} A_k \left[p + tan\left(\frac{\Phi(\xi)}{2}\right)\right]^k + \sum_{k=1}^{N} B_k \left[p + tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-k}, \tag{1.3}$$

where A_k $(0 \leqslant k \leqslant N)$ and B_k $(1 \leqslant k \leqslant N)$ are arbitrary constants to be determined, and $\Phi = \Phi(\xi)$ satisfies the following ordinary differential equation

$$\Phi'(\xi) = a\sin\Phi(\xi) + b\cos\Phi(\xi) + c, \tag{1.4}$$

The problem of solving nonlinear partial differential equations is transformed into the problem of solving nonlinear algebraic equations. A lot of calculations are involved in solving nonlinear algebraic equations. Depending on whether A_2 or B_2 are equal to 0, we divided it into three cases. By discussing and solving each equation, we got three solutions to the algebraic equations. It is shown that the improved tan $\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method can obtain a lot of exact solutions of the non-dissipative double dispersive equation, including hyperbolic function solutions, trigonometric function solutions, exponential function solutions, and rational function solutions. From the exact solutions obtained in this manuscript, we select four different types of solutions and draw 3D graphs of these solutions.

For convenience, we make a comparison and a summary. We compare the solutions in this manuscript with those in the existing literature, as shown in Table 1. We obtain some new exact solutions of the non-dissipative double dispersive equation, see (3.16),(3.18),(3.19),(3.20),(3.21),(3.28), (3.29),(3.30),(3.35),(3.36),(3.37),(3.38). These new solutions may play an important role in engineering and physics. Moreover, some of the solutions obtained in this manuscript are consistent with those in the existing literature, and some of the solutions obtained in this manuscript are more and more extensive. For example, the solution (3.25) consistent with the solution (17) in [4] when the parameter satisfies $a = \lambda, b^2 - c^2 = -4\mu, a^2 + b^2 - c^2 = \Theta$.

The outline of the manuscript is as follows: In Section 2, we give a brief description of the improved tan $\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method. In Section 3, we apply the improved tan $\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method for finding exact solutions to the non-dissipative double dispersive equation in microstructured solids. In Section 4, we plot the 3D graphs of some solutions in this manuscript with maple and compare the solutions in this manuscript with the existing solution. And finally, conclusions are given in Section 5.

2 Description of the Improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion Method

Consider a nonlinear partial differential equation of the form

$$H\left(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \cdots\right) = 0. \tag{2.1}$$

The improved tan $\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method consists of the following main steps:

Step1. Using the transformation $u = u(\xi), \xi = x - \omega t$, we reduce the Eq. (2.1) to an ordinary differential equation.

$$Q\left(u, u', -\omega u', u'', \omega^2 u'', \cdots\right) = 0. \tag{2.2}$$

Step2. Suppose that Eq. (2.2) has a formal solution

$$u(\xi) = S(\Phi) = \sum_{k=0}^{N} A_k \left[p + tan\left(\frac{\Phi(\xi)}{2}\right)\right]^k + \sum_{k=1}^{N} B_k \left[p + tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-k}, \tag{2.3}$$

where $A_k\,(0 \leqslant k \leqslant N)$ and $B_k\,(1 \leqslant k \leqslant N)$ are arbitrary constants to be determined, and $\Phi = \Phi(\xi)$ satisfies the following ordinary differential equation

$$\Phi'(\xi) = a\sin\Phi(\xi) + b\cos\Phi(\xi) + c, \tag{2.4}$$

where some special solutions of Eq. (2.4) can be seen in [25].

Step3. Substituting Eq. (2.3) and Eq. (2.4) into Eq. (2.2). Collecting the $\tan^k\left(\frac{\Phi(\xi)}{2}\right), (-N \leqslant k \leqslant N)$, then setting each coefficient to zero, we can get a set of over-determined equations for $A_k\,(0 \leqslant k \leqslant N)$, $B_k\,(1 \leqslant k \leqslant N)$, a, b, c and p.

Step4. Solving the algebraic equations in step 3, then substituting $A_0, A_1, B_1, \cdots, A_N, B_N, a, b, c, p$ in Eq. (2.3)

3 Application of Improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion method

In order to find exact solutions of Eq. (1.1), we first make the following transformations:

$$u = u(\xi), \xi = x - \omega t. \tag{3.1}$$

Substituting Eq. (3.1) into Eq. (1.1) yields

$$\varepsilon(\alpha_3 - \omega^2\alpha_4)u^{(4)} - 2\varepsilon\alpha_1(uu')' + (\omega^2 - 1)u'' = 0. \tag{3.2}$$

Integrating Eq. (3.2) twice, we get

$$\varepsilon(\alpha_3 - \omega^2\alpha_4)u'' - \varepsilon\alpha_1 u^2 + (\omega^2 - 1)u = 0. \tag{3.3}$$

Taking the homogeneous balance between u'' and u^2 in Eq. (3.3), we obtain $N = 2$.

Therefore, the solution of Eq. (3.3) takes the form

$$u(\xi) = A_0 + A_1\tan\left(\frac{\Phi(\xi)}{2}\right) + A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right) + B_1\tan^{-1}\left(\frac{\Phi(\xi)}{2}\right) + A_1\tan^{-2}\left(\frac{\Phi(\xi)}{2}\right), \tag{3.4}$$

where

$$\Phi'(\xi) = a\sin\Phi(\xi) + b\cos\Phi(\xi) + c.$$

Substituting Eq. (3.4) and Eq. (3) into Eq. (3.3), the left hand side is converted into polynomials in $\tan\left(\frac{\Phi(\xi)}{2}\right)$. Setting each coefficient of each polynomial to zero, we get a set of over-determined equations for $A_0, A_1, A_2, B_1, B_2, a, b, c, \omega$. all the coefficients of $\left(\tan\left(\frac{\Phi(\xi)}{2}\right)\right)^n$ are compared, where $n = -4, -3, -2, -1, 0, 1, 2, 3, 4$ with zero providing the following set of algebraic equations:

$$\frac{3}{2}\varepsilon(\alpha_3 - \omega^2\alpha_4)(c-b)^2 A_2 - \varepsilon\alpha_1 {A_2}^2 = 0, \tag{3.5}$$

$$\varepsilon(\alpha_3 - \omega^2\alpha_4)\left[5(c-b)A_2 + \frac{1}{2}(c-b)^2 A_1\right] - 2\varepsilon\alpha_1 A_1 A_2 = 0, \tag{3.6}$$

$$\varepsilon(\alpha_3 - \omega^2\alpha_4)\left[4(a^2 + \frac{1}{2}c^2 - \frac{1}{2}b^2)A_2 + \frac{3}{2}a(c-b)A_1\right] - \varepsilon\alpha_1 {A_1}^2 - 2\varepsilon\alpha_1 A_0 A_2 + (\omega^2 - 1)A_2 = 0, \tag{3.7}$$

$$\varepsilon(\alpha_3 - \omega^2\alpha_4)\left[3a(b+c)A_2 + (a^2 + \frac{1}{2}c^2 - \frac{1}{2}b^2)A_1\right] - 2\varepsilon\alpha_1 A_0 A_1 - 2\varepsilon\alpha_1 A_2 B_1 + (\omega^2 - 1)A_1 = 0, \tag{3.8}$$

$$\begin{aligned}&\varepsilon(\alpha_3 - \omega^2\alpha_4)\left[\frac{1}{2}(b+c)^2 A_2 + \frac{1}{2}(c-b)^2 B_2 + \frac{1}{2}a(b+c)A_1 + \frac{1}{2}a(c-b)B_1\right]\\ &-\varepsilon\alpha_1 {A_0}^2 - 2\varepsilon\alpha_1 A_1 B_1 - 2\varepsilon\alpha_1 A_2 B_2 + (\omega^2 - 1)A_0 = 0,\end{aligned} \tag{3.9}$$

$$\varepsilon(\alpha_3-\omega^2\alpha_4)\left[3a(c-b)B_2+(a^2+\frac{1}{2}c^2-\frac{1}{2}b^2)B_1\right]-2\varepsilon\alpha_1A_0B_1-2\varepsilon\alpha_1A_1B_2+(\omega^2-1)B_1=0. \quad (3.10)$$

$$\varepsilon(\alpha_3-\omega^2\alpha_4)\left[4(a^2+\frac{1}{2}c^2-\frac{1}{2}b^2)B_2+\frac{3}{2}a(b+c)B_1\right]-\varepsilon\alpha_1{B_1}^2-2\varepsilon\alpha_1A_0B_2+(\omega^2-1)B_2=0, \quad (3.11)$$

$$\varepsilon(\alpha_3-\omega^2\alpha_4)\left[5a(b+c)B_2+\frac{1}{2}(b+c)^2B_1\right]-2\varepsilon\alpha_1B_1B_2=0, \quad (3.12)$$

$$\frac{3}{2}(b+c)^2\varepsilon(\alpha_3-\omega^2\alpha_4)B_2-\varepsilon\alpha_1{B_2}^2=0, \quad (3.13)$$

By solving the set of over-determined equations Eq. (3.5)–Eq. (3.13), we obtained the following sets of non-trivial solutions.

Case1.

$$\begin{aligned}&a=a,b=b,c=c,\Delta=a^2+b^2-c^2,l=\pm1,\bar{\xi}=\xi+\xi_0,\omega=\pm\sqrt{\frac{1+l\varepsilon\alpha_3\Delta}{1+l\varepsilon\alpha_4\Delta}},\\&A_0=\frac{(\alpha_3-\alpha_4)(a^2-2b^2+2c^2+l\Delta)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)},A_1=\frac{3(\alpha_3-\alpha_4)a(c-b)}{\alpha_1(1+l\varepsilon\alpha_4\Delta)},A_2=\frac{3(\alpha_3-\alpha_4)(c-b)^2}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)},\\&B_1=0,B_2=0,u(\xi)=A_0+\alpha_1\tan\left(\frac{\Phi(\xi)}{2}\right)+A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right).\end{aligned}$$

where ξ_0 is arbitrary constant. With 19 kinds of results of reference [25], we obtain solutions of equation Eq. (1.1):

When $\Delta<0$ and $b-c\neq0$, then the following trigonometric functional solutions are obtained:

$$\begin{aligned}u_1(\xi)=&\frac{(\alpha_3-\alpha_4)(a^2-2b^2+2c^2+l\Delta)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)}-\frac{3(\alpha_3-\alpha_4)a}{\alpha_1(1+l\varepsilon\alpha_4\Delta)}\left(a-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right)\\&+\frac{3(\alpha_3-\alpha_4)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)}\left(a-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right)^2.\end{aligned} \quad (3.14)$$

When $\Delta>0$ and $b-c\neq0$, then the following hyperbolic functional solutions are obtained:

$$\begin{aligned}u_2(\xi)=&\frac{(\alpha_3-\alpha_4)(a^2-2b^2+2c^2+l\Delta)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)}-\frac{3(\alpha_3-\alpha_4)a}{\alpha_1(1+l\varepsilon\alpha_4\Delta)}\left(a+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right)\\&+\frac{3(\alpha_3-\alpha_4)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)}\left(a+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right)^2.\end{aligned} \quad (3.15)$$

When $a=0,c=0$, then the following trigonometric functional solutions are obtained:

$$u_3(\xi)=\frac{(\alpha_3-\alpha_4)(-2b^2+lb^2)}{2\alpha_1(1+l\varepsilon\alpha_4b^2)}+\frac{3(\alpha_3-\alpha_4)b^2}{2\alpha_1(1+l\varepsilon\alpha_4b^2)}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1},\frac{2e^{b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}\right]\right). \quad (3.16)$$

When $\Delta=0$, then the following rational functional solutions are obtained:

$$u_4(\xi)=\frac{3(\alpha_3-\alpha_4)a^2}{2\alpha_1}-\frac{3(\alpha_3-\alpha_4)a}{\alpha_1}\left(\frac{a\bar{\xi}+2}{\bar{\xi}}\right)+\frac{3(\alpha_3-\alpha_4)}{2\alpha_1}\left(\frac{a\bar{\xi}+2}{\bar{\xi}}\right)^2. \quad (3.17)$$

When $a=c=ka,b=-ka$, then the following exponential functional solutions are obtained:

$$u_5(\xi)=\frac{(\alpha_3-\alpha_4)(k^2a^2+lk^2a^2)}{2\alpha_1(1+l\varepsilon\alpha_4k^2a^2)}+\frac{6(\alpha_3-\alpha_4)k^2a^2}{\alpha_1(1+l\varepsilon\alpha_4k^2a^2)}\left(\frac{e^{ka\bar{\xi}}}{1-e^{ka\bar{\xi}}}+\left(\frac{e^{ka\bar{\xi}}}{1-e^{ka\bar{\xi}}}\right)^2\right). \quad (3.18)$$

When $c = a$, then the following exponential functional solutions are obtained:

$$u_6(\xi) = \frac{(\alpha_3-\alpha_4)(3a^2-2b^2+lb^2)}{2\alpha_1(1+l\varepsilon\alpha_4 b^2)} - \frac{3(\alpha_3-\alpha_4)a(a-b)}{\alpha_1(1+l\varepsilon\alpha_4 b^2)}\left(\frac{(a+b)e^{b\bar{\xi}}-1}{(a-b)e^{b\bar{\xi}}-1}\right) + \frac{3(\alpha_3-\alpha_4)(a-b)^2}{2\alpha_1(1+l\varepsilon\alpha_4 b^2)}\left(\frac{(a+b)e^{b\bar{\xi}}-1}{(a-b)e^{b\bar{\xi}}-1}\right)^2. \tag{3.19}$$

When $a = c$, then the following exponential functional solutions are obtained:

$$u_7(\xi) = \frac{(\alpha_3-\alpha_4)(3c^2-2b^2+lb^2)}{2\alpha_1(1+l\varepsilon\alpha_4 b^2)} + \frac{3(\alpha_3-\alpha_4)c(c-b)}{\alpha_1(1+l\varepsilon\alpha_4 b^2)}\left(\frac{(b+c)e^{b\bar{\xi}}+1}{(b-c)e^{b\bar{\xi}}-1}\right) + \frac{3(\alpha_3-\alpha_4)(c-b)^2}{2\alpha_1(1+l\varepsilon\alpha_4 b^2)}\left(\frac{(b+c)e^{b\bar{\xi}}+1}{(b-c)e^{b\bar{\xi}}-1}\right)^2. \tag{3.20}$$

When $c = -a$, then the following exponential functional solutions are obtained:

$$u_8(\xi) = \frac{(\alpha_3-\alpha_4)(3a^2-2b^2+lb^2)}{2\alpha_1(1+l\varepsilon\alpha_4 b^2)} - \frac{3(\alpha_3-\alpha_4)a(a+b)}{\alpha_1(1+l\varepsilon\alpha_4 b^2)}\left(\frac{e^{b\bar{\xi}}+b-a}{e^{b\bar{\xi}}-b-a}\right) + \frac{3(\alpha_3-\alpha_4)(a+b)^2}{2\alpha_1(1+l\varepsilon\alpha_4 b^2)}\left(\frac{e^{b\bar{\xi}}+b-a}{e^{b\bar{\xi}}-b-a}\right)^2. \tag{3.21}$$

When $b = -c$, then the following exponential functional solutions are obtained:

$$u_9(\xi) = \frac{(\alpha_3-\alpha_4)(a^2+la^2)}{2\alpha_1(1+l\varepsilon\alpha_4 a^2)} - \frac{6(\alpha_3-\alpha_4)ac}{\alpha_1(1+l\varepsilon\alpha_4 a^2)}\left(\frac{ae^{a\bar{\xi}}}{ce^{a\bar{\xi}}-1}\right) + \frac{6(\alpha_3-\alpha_4)c^2}{\alpha_1(1+l\varepsilon\alpha_4 a^2)}\left(\frac{ae^{a\bar{\xi}}}{ce^{a\bar{\xi}}-1}\right)^2. \tag{3.22}$$

When $a = b = 0$, then the following trigonometric functional solutions are obtained:

$$u_{10}(\xi) = \frac{(\alpha_3-\alpha_4)(2c^2-lc^2)}{2\alpha_1(1-l\varepsilon\alpha_4 c^2)} + \frac{3(\alpha_3-\alpha_4)c^2}{2\alpha_1(1-l\varepsilon\alpha_4 c^2)}\tan^2\left(\frac{c\bar{\xi}}{2}\right). \tag{3.23}$$

Case2.

$$a = a, b = b, c = c, \Delta = a^2+b^2-c^2, l = \pm 1, \bar{\xi} = \xi + \xi_0, \omega = \omega = \pm\sqrt{\frac{1+l\varepsilon\alpha_3\Delta}{1+l\varepsilon\alpha_4\Delta}},$$

$$A_0 = \frac{(\alpha_3-\alpha_4)(a^2-2b^2+2c^2+l\Delta)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)}, A_1 = 0, A_2 = 0, B_1 = \frac{3(\alpha_3-\alpha_4)}{\alpha_1(1+l\varepsilon\alpha_4\Delta)}a(b+c),$$

$$B_2 = \frac{3(\alpha_3-\alpha_4)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)}(b+c)^2, u(\xi) = A_0 + B_1\tan^{-1}\left(\frac{\Phi(\xi)}{2}\right) + B_2\tan^{-2}\left(\frac{\Phi(\xi)}{2}\right).$$

where ξ_0 is arbitrary constant. With 19 kinds of results of reference [25], we obtain solutions of equation Eq. (1.1):

When $\Delta < 0$ and $b - c \neq 0$, then the following trigonometric functional solutions are obtained:

$$u_{11}(\xi) = \frac{(\alpha_3-\alpha_4)(a^2-2b^2+2c^2+l\Delta)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)} + \frac{3(\alpha_3-\alpha_4)a(b^2-c^2)}{\alpha_1(1+l\varepsilon\alpha_4\Delta)}\left(a - \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right)^{-1} + \frac{3(\alpha_3-\alpha_4)(b^2-c^2)^2}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)}\left(a - \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right)^{-2}. \tag{3.24}$$

When $\Delta > 0$ and $b - c \neq 0$, then the following hyperbolic functional solutions are obtained:

$$u_{12}(\xi) = \frac{(\alpha_3-\alpha_4)(a^2-2b^2+2c^2+l\Delta)}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)} + \frac{3(\alpha_3-\alpha_4)a(b^2-c^2)}{\alpha_1(1+l\varepsilon\alpha_4\Delta)}\left(a + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right)^{-1} + \frac{3(\alpha_3-\alpha_4)(b^2-c^2)^2}{2\alpha_1(1+l\varepsilon\alpha_4\Delta)}\left(a + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right)^{-2}. \tag{3.25}$$

When $\Delta = 0$, then the following rational functional solutions are obtained:

$$u_{13}(\xi) = \frac{3(\alpha_3 - \alpha_4)a^2}{2\alpha_1} - \frac{3(\alpha_3 - \alpha_4)a^3}{\alpha_1}\left(\frac{a\bar{\xi}+2}{\bar{\xi}}\right)^{-1} + \frac{3(\alpha_3 - \alpha_4)a^4}{2\alpha_1}\left(\frac{a\bar{\xi}+2}{\bar{\xi}}\right)^{-2}. \tag{3.26}$$

When $a = b = c = ka$, then the following exponential functional solutions are obtained:

$$u_{14}(\xi) = \frac{(\alpha_3 - \alpha_4)(k^2a^2 + lk^2a^2)}{2\alpha_1(1 + l\varepsilon\alpha_4 k^2a^2)} + \frac{6(\alpha_3 - \alpha_4)k^2a^2}{\alpha_1(1 + l\varepsilon\alpha_4 k^2a^2)}\left(\left(e^{ka\bar{\xi}} - 1\right)^{-1} + \left(e^{ka\bar{\xi}} - 1\right)^{-2}\right). \tag{3.27}$$

When $c = a$, then the following exponential functional solutions are obtained:

$$\begin{aligned} u_{15}(\xi) = {} & \frac{(\alpha_3 - \alpha_4)(3a^2 - 2b^2 + lb^2)}{2\alpha_1(1 + l\varepsilon\alpha_4 b^2)} - \frac{3(\alpha_3 - \alpha_4)a(a+b)}{\alpha_1(1 + l\varepsilon\alpha_4 b^2)}\left(\frac{(a+b)e^{b\bar{\xi}} - 1}{(a-b)e^{b\bar{\xi}} - 1}\right)^{-1} \\ & + \frac{3(\alpha_3 - \alpha_4)(a+b)^2}{2\alpha_1(1 + l\varepsilon\alpha_4 b^2)}\left(\frac{(a+b)e^{b\bar{\xi}} - 1}{(a-b)e^{b\bar{\xi}} - 1}\right)^{-2}. \end{aligned} \tag{3.28}$$

When $a = c$, then the following exponential functional solutions are obtained:

$$\begin{aligned} u_{16}(\xi) = {} & \frac{(\alpha_3 - \alpha_4)(3c^2 - 2b^2 + lb^2)}{2\alpha_1(1 + l\varepsilon\alpha_4 b^2)} + \frac{3(\alpha_3 - \alpha_4)c(b+c)}{\alpha_1(1 + l\varepsilon\alpha_4 b^2)}\left(\frac{(b+c)e^{b\bar{\xi}} + 1}{(b-c)e^{b\bar{\xi}} - 1}\right)^{-1} \\ & + \frac{3(\alpha_3 - \alpha_4)(b+c)^2}{2\alpha_1(1 + l\varepsilon\alpha_4 b^2)}\left(\frac{(b+c)e^{b\bar{\xi}} + 1}{(b-c)e^{b\bar{\xi}} - 1}\right)^{-2}. \end{aligned} \tag{3.29}$$

When $c = -a$, then the following exponential functional solutions are obtained:

$$\begin{aligned} u_{17}(\xi) = {} & \frac{(\alpha_3 - \alpha_4)(3a^2 - 2b^2 + lb^2)}{2\alpha_1(1 + l\varepsilon\alpha_4 b^2)} + \frac{3(\alpha_3 - \alpha_4)a(b-a)}{\alpha_1(1 + l\varepsilon\alpha_4 b^2)}\left(\frac{e^{b\bar{\xi}} + b - a}{e^{b\bar{\xi}} - b - a}\right)^{-1} \\ & + \frac{3(\alpha_3 - \alpha_4)(b-a)^2}{2\alpha_1(1 + l\varepsilon\alpha_4 b^2)}\left(\frac{e^{b\bar{\xi}} + b - a}{e^{b\bar{\xi}} - b - a}\right)^{-2}. \end{aligned} \tag{3.30}$$

When $a = b = 0$, then the following trigonometric functional solutions are obtained:

$$u_{18}(\xi) = \frac{(\alpha_3 - \alpha_4)(2c^2 - lc^2)}{2\alpha_1(1 - l\varepsilon\alpha_4 c^2)} + \frac{3(\alpha_3 - \alpha_4)c^2}{2\alpha_1(1 - l\varepsilon\alpha_4 c^2)}\tan^{-2}\left(\frac{c\bar{\xi}}{2}\right). \tag{3.31}$$

When $b = c$, then the following rational functional solutions are obtained:

$$u_{19}(\xi) = \frac{(\alpha_3 - \alpha_4)(a^2 + la^2)}{2\alpha_1(1 + l\varepsilon\alpha_4 a^2)} + \frac{6(\alpha_3 - \alpha_4)a^2c}{\alpha_1(1 + l\varepsilon\alpha_4 a^2)}\left(e^{a\bar{\xi}} - c\right)^{-1} + \frac{6(\alpha_3 - \alpha_4)a^2c^2}{\alpha_1(1 + l\varepsilon\alpha_4 a^2)}\left(e^{a\bar{\xi}} - c\right)^{-2}. \tag{3.32}$$

Case3.

$$\begin{aligned} & a = 0, b = b, c = c, l = \pm 1, \bar{\xi} = \xi + \xi_0, \omega = \pm\sqrt{\frac{1 + 4l\varepsilon\alpha_3(c^2 - b^2)}{1 + 4l\varepsilon\alpha_4(c^2 - b^2)}}, A_1 = 0, B_1 = 0, \\ & A_0 = \frac{(1 + 2l)(\alpha_3 - \alpha_4)(c^2 - b^2)}{\alpha_1(1 + 4l\varepsilon\alpha_4(c^2 - b^2))}, A_2 = \frac{3(\alpha_3 - \alpha_4)(c-b)^2}{2\alpha_1(1 + 4l\varepsilon\alpha_4(c^2 - b^2))}, B_2 = \frac{3(\alpha_3 - \alpha_4)(b+c)^2}{2\alpha_1(1 + 4l\varepsilon\alpha_4(c^2 - b^2))}, \\ & u(\xi) = A_0 + A_2\tan^2\left(\frac{\Phi(\xi)}{2}\right) + B_2\tan^{-2}\left(\frac{\Phi(\xi)}{2}\right). \end{aligned}$$

where ξ_0 is arbitrary constant. With 19 kinds of results of reference [25], we obtain solutions of equation Eq. (1.1):

When $\Delta < 0$ and $b - c \neq 0$, then the following trigonometric functional solutions are obtained:

$$\begin{aligned} u_{20}(\xi) = & \frac{(1+2l)(\alpha_3-\alpha_4)(c^2-b^2)}{\alpha_1(1+4l\varepsilon\alpha_4(c^2-b^2))} + \frac{3(\alpha_3-\alpha_4)(c^2-b^2)}{2\alpha_1(1+4l\varepsilon\alpha_4(c^2-b^2))}\tan^2\left(\frac{\sqrt{c^2-b^2}}{2}\bar{\xi}\right) \\ & + \frac{3(\alpha_3-\alpha_4)(c^2-b^2)}{2\alpha_1(1+4l\varepsilon\alpha_4(c^2-b^2))}\tan^{-2}\left(\frac{\sqrt{c^2-b^2}}{2}\bar{\xi}\right). \end{aligned} \tag{3.33}$$

When $\Delta > 0$ and $b - c \neq 0$, then the following hyperbolic functional solutions are obtained:

$$\begin{aligned} u_{21}(\xi) = & \frac{(1+2l)(\alpha_3-\alpha_4)(c^2-b^2)}{\alpha_1(1+4l\varepsilon\alpha_4(c^2-b^2))} + \frac{3(\alpha_3-\alpha_4)(b^2-c^2)}{2\alpha_1(1+4l\varepsilon\alpha_4(c^2-b^2))}\tanh^2\left(\frac{\sqrt{b^2-c^2}}{2}\bar{\xi}\right) \\ & + \frac{3(\alpha_3-\alpha_4)(b^2-c^2)}{2\alpha_1(1+4l\varepsilon\alpha_4(c^2-b^2))}\tanh^{-2}\left(\frac{\sqrt{b^2-c^2}}{2}\bar{\xi}\right). \end{aligned} \tag{3.34}$$

When $a = 0, c = 0$, then the following trigonometric functional solutions are obtained:

$$\begin{aligned} u_{22}(\xi) = & \frac{(-1-2l)(\alpha_3-\alpha_4)b^2}{\alpha_1(1-4l\varepsilon\alpha_4 b^2)} + \frac{3(\alpha_3-\alpha_4)b^2}{2\alpha_1(1-4l\varepsilon\alpha_4 b^2)}\tan^2\left(\frac{1}{2}\arctan\left[\frac{e^{2b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}, \frac{2e^{b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}\right]\right) \\ & + \frac{3(\alpha_3-\alpha_4)b^2}{2\alpha_1(1-4l\varepsilon\alpha_4 b^2)}\tan^{-2}\left(\frac{1}{2}\arctan\left[\frac{e^{2b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}, \frac{2e^{b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}\right]\right). \end{aligned} \tag{3.35}$$

When $c = a$, then the following exponential functional solutions are obtained:

$$u_{23}(\xi) = \frac{(-1-2l)(\alpha_3-\alpha_4)b^2}{\alpha_1(1-4l\varepsilon\alpha_4 b^2)} + \frac{3(\alpha_3-\alpha_4)b^2}{2\alpha_1(1-4l\varepsilon\alpha_4 b^2)}\left[\left(\frac{be^{b\bar{\xi}}-1}{-be^{b\bar{\xi}}-1}\right)^2 + \left(\frac{be^{b\bar{\xi}}-1}{-be^{b\bar{\xi}}-1}\right)^{-2}\right]. \tag{3.36}$$

When $a = c$, then the following exponential functional solutions are obtained:

$$u_{24}(\xi) = \frac{(-1-2l)(\alpha_3-\alpha_4)b^2}{\alpha_1(1-4l\varepsilon\alpha_4 b^2)} + \frac{3(\alpha_3-\alpha_4)b^2}{2\alpha_1(1-4l\varepsilon\alpha_4 b^2)}\left[\left(\frac{be^{b\bar{\xi}}+1}{be^{b\bar{\xi}}-1}\right)^2 + \left(\frac{be^{b\bar{\xi}}+1}{be^{b\bar{\xi}}-1}\right)^{-2}\right]. \tag{3.37}$$

When $c = -a$, then the following exponential functional solutions are obtained:

$$u_{25}(\xi) = \frac{(-1-2l)(\alpha_3-\alpha_4)b^2}{\alpha_1(1-4l\varepsilon\alpha_4 b^2)} + \frac{3(\alpha_3-\alpha_4)b^2}{2\alpha_1(1-4l\varepsilon\alpha_4 b^2)}\left[\left(\frac{e^{b\bar{\xi}}+b}{e^{b\bar{\xi}}-b}\right)^2 + \left(\frac{e^{b\bar{\xi}}+b}{e^{b\bar{\xi}}-b}\right)^{-2}\right]. \tag{3.38}$$

When $a = b = 0$, then the following trigonometric functional solutions are obtained:

$$u_{26}(\xi) = \frac{(1+2l)(\alpha_3-\alpha_4)c^2}{\alpha_1(1+4l\varepsilon\alpha_4 c^2)} + \frac{3(\alpha_3-\alpha_4)c^2}{2\alpha_1(1+4l\varepsilon\alpha_4 c^2)}\left[\tan^2\left(\frac{c\bar{\xi}}{2}\right) + \tan^{-2}\left(\frac{c\bar{\xi}}{2}\right)\right]. \tag{3.39}$$

4 Analysis and Discussion

In this section, we draw some images of the solutions and compare the solutions obtained in this manuscript with some known solutions.

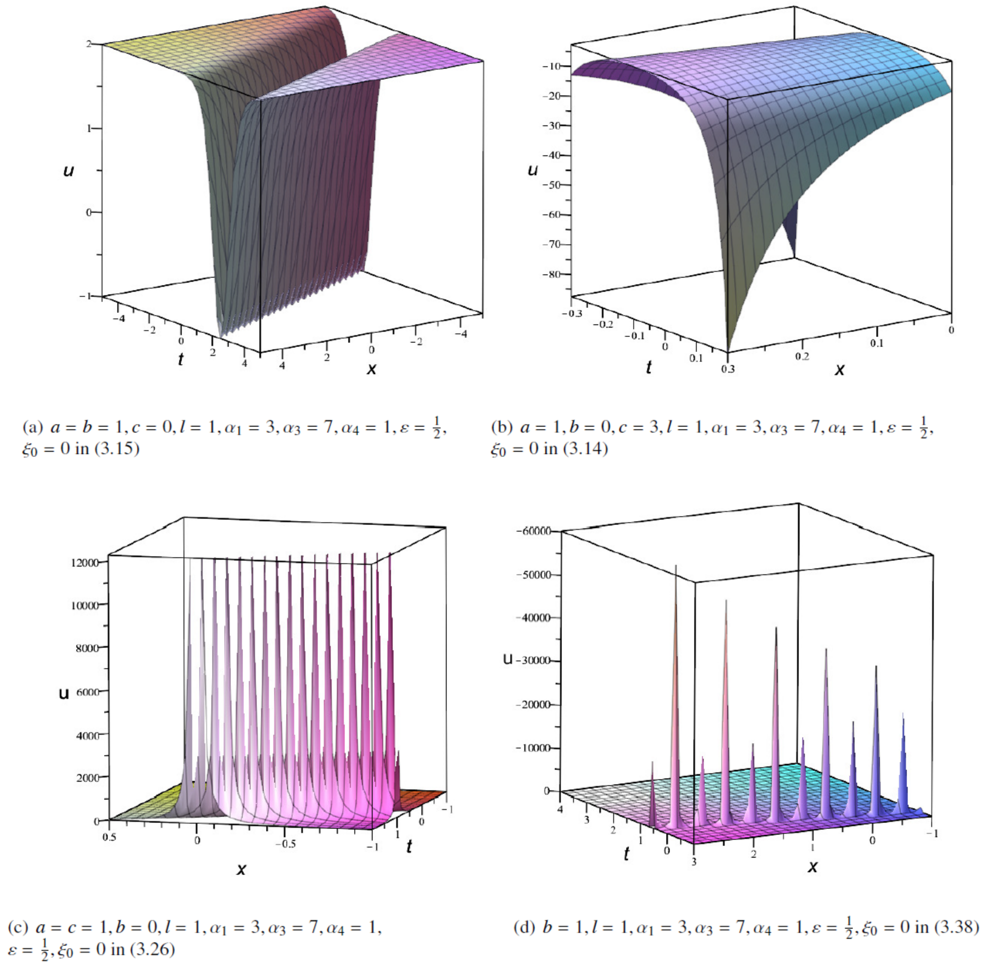
Firstly, we select four different types of solutions from the obtained solutions and draw 3D graphs of these solutions. Some exact solutions are shown numerically in Fig. 1a - 1d.

Fig 1a. Hyperbolic functional solution $u_2(\xi)$ for the values of parameters $a = b = 1, c = 0, l = 1, \alpha_1 = 3, \alpha_3 = 7, \alpha_4 = 1, \varepsilon = \frac{1}{2}, \xi_0 = 0$.

Fig 1b. Trigonometric functional solution $u_1(\xi)$ for the values of parameters $a = 1, b = 0, c = 3, l = 1, \alpha_1 = 3, \alpha_3 = 7, \alpha_4 = 1, \varepsilon = \frac{1}{2}, \xi_0 = 0$.

Fig 1c. Rational functional solution $u_{13}(\xi)$ for the values of parameters $a = c = 1, b = 0, l = 1, \alpha_1 = 3, \alpha_3 = 7, \alpha_4 = 1, \varepsilon = \frac{1}{2}, \xi_0 = 0$.

Fig 1d. Exponential functional solution $u_{25}(\xi)$ for the values of parameters $b = 1, l = 1, \alpha_1 = 3, \alpha_3 = 7, \alpha_4 = 1, \varepsilon = \frac{1}{2}, \xi_0 = 0$.



(a) $a = b = 1, c = 0, l = 1, \alpha_1 = 3, \alpha_3 = 7, \alpha_4 = 1, \varepsilon = \frac{1}{2}$, $\xi_0 = 0$ in (3.15)

(b) $a = 1, b = 0, c = 3, l = 1, \alpha_1 = 3, \alpha_3 = 7, \alpha_4 = 1, \varepsilon = \frac{1}{2}$, $\xi_0 = 0$ in (3.14)

(c) $a = c = 1, b = 0, l = 1, \alpha_1 = 3, \alpha_3 = 7, \alpha_4 = 1$, $\varepsilon = \frac{1}{2}, \xi_0 = 0$ in (3.26)

(d) $b = 1, l = 1, \alpha_1 = 3, \alpha_3 = 7, \alpha_4 = 1, \varepsilon = \frac{1}{2}, \xi_0 = 0$ in (3.38)

Fig. 1. Exact solutions of the generalized Pochhammer-Chree equation when $n = 1$

We compared the solutions obtained in this manuscript with the existing solution, as shown in Table 1. In Table 1, the first column shows the different values of parameters such as a, b, c and l, the second column shows the equation number of this manuscript, and the third column shows the equation number in the existing literature. It can be found that some of the solutions obtained in this manuscript consistent with existing solutions, which shows the effectiveness of the improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method. For example, the solution (3.25) consistent with the solution (17) in [4] when the parameter satisfies $a = \lambda, b^2 - c^2 = -4\mu, a^2 + b^2 - c^2 = \Theta$. What's more, the improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method provides a lot of new solutions with additional free parameters, see (3.16),(3.18),(3.19),(3.20),(3.21), (3.28),(3.29),(3.30),(3.35),(3.36),(3.37),(3.38). The limitation of this method is that the traveling wave solution of this equation is obtained. And we have assumed the form of the solution of this equation in advance. We will continue to study other types of solutions to this equation in the future.

Table 1. Comparison of solutions obtained in this manuscript with already existing solutions

Parametric values	Solutions obtained in this manuscript	already existing solutions
$a = \lambda, b^2 - c^2 = -4\mu, a^2 + b^2 - c^2 = \Theta$	(3.25),(3.24),(3.27) , (3.26)	**(17),(18),(19),(20)**[4]
$a = 0, b = c$	(3.17)	**(21)**[4]
$a = 1, c = -\frac{1}{d}$	(3.32)	**(12),(13)**[6]
$A = 0, k = 1, \lambda = 1, \delta_2 = \frac{3}{4}(c^2 - b^2), a = 0$	(3.25)	**(10)[10]**
$A = 0, k = 1, \lambda = 1, c = 2\sqrt{\frac{\delta_2}{3}}$	(3.31)	**(11)[10]**
$a = c_1, c = \pm c_2, l = -1$	(3.22)	**(18),(19)[14]**
$a = 0, b = c_0 - c_2, c = c_0 + c_2, l = 1$	(3.24),(3.25)	**(21),(22)**[14]
$a = 0, b = c_0 - c_2, c = c_0 + c_2, l = -1$	(3.14),(3.15)	**(24),(25)**[14]
$b = c_2 - c_0, c = -c_0 - c_2, l = -1$	(3.33),(3.34)	**(27),(28)**[14]
$c = 2, l = -1$	(3.39)	**(43)**[14]
$b^2 - c^2 = 4, l = 1$	(3.34)	**(22)**[15]
$c = 2$	(3.23)	**(43) [16]**
...	(3.16),(3.18),(3.19),(3.20),(3.21),(3.28) ,(3.29),(3.30),(3.35) ,(3.36),(3.37),(3.38)	**No solution found corresponding to this solutions**

5 Conclusion

In this manuscript, the improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method is applied to derive traveling wave solutions to the non-dissipative double dispersive equation in microstructured solids. Abundant exact traveling wave solutions are obtained, including hyperbolic function solutions, trigonometric function solutions, exponential solutions, and rational solutions. It is worth noting that we got some new solutions, and some solutions are consistent with already published results. Seeking the exact solution of a nonlinear partial differential equation with a physical background is helpful to understand the motion law of matter and to explain the corresponding physical phenomena scientifically. In the future, we will continue to explore its physical significance from the new analytical solution we have obtained. Moreover, we will try to solve more partial differential equations that arise in engineering, mathematical physics, and other scientific real-time application fields by the improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ -expansion method.

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Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

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Competing Interests

The authors declare that no competing interests exist.

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