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Strong Convergence Theorems for Asymptotically Pseudocontractive Mappings in the Intermediate Sense

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Research Article

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Abstract

In this study, we prove a strong convergence of Noor type scheme for a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense without assuming any form of compactness. Consequently, we also obtain a convergence result for the class of asymptotically strict pseudocontractive mappings in the intermediate sense. Our results are improvements and extensions of some of the results in literature.

Keywords: Strong convergence; asymptotically nonexpansive mappings; asymptotically pseudocontractive mappings in the intermediate sense; asymptotically strict pseudocontractive mappings in the intermediate sense;

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1 Introduction

In the sequel, we give the following definitions of some of the concepts that will feature prominently in this study.

Definition 1.1. Let $T : C \to C$ be a mapping. T is said to be
(1)*asymptotically nonexpansive* (Sahu *et al.* (2009)) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \forall n \geq 1, x, y \in C. \tag{1.1}$$

Goebel and Kirk (1972) introduced the class of asymptotically nonexpansive mappings as a generalization of the class of nonexpansive mappings.

(2) *asymptotically nonexpansive in the intermediate sense* (Zegeye *et al.* (2011)) if it is continuous and the following inequality holds:

$$\lim_{n\to\infty} \sup \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.2}$$

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Observe that if we define

$$\zeta_n = \max\left\{0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|)\right\}, \tag{1.3}$$

then $\zeta_n \to 0$ as $n \to \infty$. Hence, (1.2) can be reduced to

$$\|T^n x - T^n y\| \leq \|x-y\| + \zeta_n, \ \ \forall n \geq 1, x, y \in C. \tag{1.4}$$

The class of asymptotically nonexpansive mapping in the intermediate sense was introduced in 1993 by Bruck *et al.* (1993). We remark that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

(3) *strict pseudocontractive* (Qin *et al.* (2010)) if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{1.5}$$

The class of strict pseudocontractive maps was introduced in 1967 by Browder and Petryshyn (1967). Marino and Xu (2007) established that the fixed point of strict pseudocontractions is closed convex and they obtained a weak convergence theorem for strictly pseudocontractive mappings by Mann iterative process.

(4) *asymptotically strict pseudocontractive* (Zegeye *et al.* (2011)) if there exists a constant $k \in [0,1)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leq k_n\|x-y\|^2 + k\|(I-T^n)x - (I-T^n)y\|^2, \quad \forall x, y \in C. \tag{1.6}$$

The class of asymptotically strict pseudocontractive mappings was introduced by Liu (1996). We remark that the class of asymptotically strict pseudocontractive mappings is a generalization of the class of strict pseudocontractive mappings.

(5) *asymptotically strict pseudocontractive in the intermediate sense* (Qin *et al.* (2010)) if there exist a constant $k \in [0,1)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \sup \sup_{x,y\in C}(\|T^n x - T^n y\|^2 - k_n\|x-y\|^2 - k\|(I-T^n)x - (I-T^n)y\|^2) \leq 0. \tag{1.7}$$

Put

$$\zeta_n = \max\left\{0, \sup_{x,y\in C}(\|T^n x - T^n y\|^2 - k_n\|x-y\|^2 - k\|(I-T^n)x - (I-T^n)y\|^2)\right\}. \tag{1.8}$$

It follows that $\zeta_n \to 0$ as $n \to \infty$. Then, (1.7) is reduced to the following:

$$\|T^n x - T^n y\|^2 \leq k_n\|x-y\|^2 + k\|(I-T^n)x - (I-T^n)y\|^2 + \zeta_n, \forall n \geq 1, x, y \in C. \tag{1.9}$$

Sahu *et al.* (2009) introduced the class of asymptotically strict pseudocontractive mappings in the intermediate sense. Zhao and He (2010) obtained some weak and strong convergence results for this class of nonlinear maps. We remark that if $\zeta_n = 0 \ \ \forall n \geq 1$ in (1.9), then we obtain (1.6), meaning that the class of asymptotically strict pseudocontractive mappings in the intermediate sense contains properly the class of asymptotically strict pseudocontractive mappings.

(6) *pseudocontractive* (Qin *et al.* (2010)) if for any $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2, \tag{1.10}$$

and it is well known that condition (1.10) is equivalent to the following:

$$\|x-y\| \leq \|x-y+s[(I-Tx)-(I-Ty)]\|, \forall s > 0, x, y \in C, \tag{1.11}$$

(7) *asymptotically pseudocontractive* (Qin *et al.* (2010)) if there exists a sequence

$\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2, \ \ \forall n \geq 1, x, y \in C. \tag{1.12}$$

Observe that (1.12) is equivalent to

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|x - y - (T^n x - T^n y)\|^2, \forall n \geq 1, x, y \in C. \tag{1.13}$$

The class of asymptotically pseudocontractive mapping was introduced in 1991 by Schu (1991). Rhoades (1976) produced an example to show that the class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings.

(8) *asymptotically pseudocontractive mapping in the intermediate sense* (Qin *et al.* 2010)) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \sup \sup_{x,y\in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \leq 0. \tag{1.14}$$

Put

$$\tau_n = \max \left\{ 0, \sup_{x,y\in C} (\langle T^n x - T^n y, x - y \rangle - k_n \|x - y\|^2) \right\}. \tag{1.15}$$

It follows that $\tau_n \to 0$ as $n \to \infty$. Hence, (1.14) is reduced to the following:

$$\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2 + \tau_n, \forall n \geq 1, x, y \in C. \tag{1.16}$$

In real Hilbert spaces, we observe that (1.16) is equivalent to

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|(I - T^n)x - (I - T^n)y\|^2 + 2\tau_n, \forall n \geq 1, x, y \in C. \tag{1.17}$$

X. Qin *et al.* (2010) introduced the class of asymptotically pseudocontractive mappings in the intermediate sense. We remark that if $\tau_n = 0 \ \ \forall n \geq 1$, then the class of asymptotically pseudocontractive mappings in the intermediate sense is reduced to the class of asymptotically pseudocontractive mappings.

X. Qin *et al.* (2010) proved the following theorem.

Theorem QCK. Let H be a real Hilbert space, $C \subset H$ be nonempty closed bounded and convex. Let T be a uniformly L-Lipschitzian and asymptotically pseudocontractive self-map of C in the intermediate sense with sequences $\{k_n\} \subset [1, \infty)$ and $\{\tau_n\} \subset [0, \infty)$ defined as in (1.17). Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$\begin{cases} y_n = \beta_n T^n x_n + (1 - \beta_n)x_n \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{cases} \tag{1.18}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \tau_n < \infty, \ \sum_{n=1}^{\infty} (q_n^2 - 1) < \infty$ where $q_n := 2k_n - 1$ for each $n \geq 1$;
(ii) $a \leq \alpha_n \leq \beta_n \leq b$ for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$.
Then the sequence $\{x_n\}$ generated by (1.18) converges weakly to a fixed point of T.

Zegeye *et al.* (2011) proved a strong convergence theorem of Ishikawa type scheme (1.18) for the class of asymptotically pseudocontractive mappings in the intermediate sense without the use of the hybrid method adopted by X. Qin *et al.* (2010).

Theorem ZRC. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\} \subset [1, \infty)$ and $\{\tau_n\} \subset [0, \infty)$ defined as in (1.14). Assume that the interior of $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$\begin{cases} y_n = \beta_n T^n x_n + (1 - \beta_n)x_n \\ \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1, \end{cases} \tag{1.19}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \tau_n < \infty, \ \sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$ where $q_n := 2k_n - 1$ for each $n \geq 1;$
(ii) $a \leq \alpha_n \leq \beta_n \leq b$ for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1 + L^2} - 1])$.
Then the sequence $\{x_n\}$ generated by (1.19) converges strongly to a fixed point of T.

Noor *et al.* (2001) gave a three-step iteration process for solving non-linear operator equations in real Banach spaces.
Consider the following Noor iteration scheme: Let $T : C \to C$ be a mapping. For an arbitrary $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty} \subset C$ defined by

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n T z_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_n T x_n, \ \ n \geq 0, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n \end{aligned} \tag{1.20}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, are three sequences satisfying $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ for each n.
Olaleru and Mogbademu (2011) and (2012) obtained some convergence results for the modified Noor iterative scheme introduced by Rafiq (2006). Zhou (2009) introduced a new three-step iterative scheme with errors.

It was established by Bnouhachem *et al.* (2006) that three-step method performs better than two-step and one-step methods for solving variational inequalities. Glowinski and P. Le Tallec in 1989 applied three-step iterative sequences for finding the approximate solutions of the elastoviscoplasticity problem, eigenvalue problems and in the liquid crystal theory. Moreover, three-step schemes are natural generalization of the splitting methods to solve partial differential equations, (see Qihou (2002), Senter and Dotson (1974), Shahzad and Udomene (2006), Suantai (2005)). What this means is that Noor three-step methods are at times robust and more efficient than the Mann (one-step) and Ishikawa (two-step) type schemes for solving problems of nonlinear equations.

The following question is natural:
Is it possible to obtain a strong convergence of Noor type scheme (1.20) to a fixed point of asymptotically pseudocontractive mappings in the intermediate sense?

We give the following definitions and lemmas which will be useful in this study.

The folowing function was studied by Alber (1996), Kamimula and Takahashi (2002) and Reich (1996). Let $\phi : H \times H \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \quad \text{for any } x, y \in H. \tag{1.21}$$

From the definition of ϕ, we observe that:

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \text{for any } x, y \in H. \tag{1.22}$$

The function ϕ has the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z - y, x - z\rangle \quad \text{for all } x, y, z \in H. \tag{1.23}$$

Lemma 1.2. (Zegeye *et al.* (2011)) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 + \gamma_n)a_n + \sigma_n, \quad n \geq n_0, \tag{1.24}$$

where, n_0 is some nonnegative integer. If $\sum \gamma_n < \infty$ and $\sum |\sigma_n| < \infty$. Then, $\lim_{n\to\infty} a_n$ exists.

Lemma 1.3. (Zegeye *et al.* (2011)) Let H be a real Hilbert space. Then the following equality holds:

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{1.25}$$

for all $\alpha \in (0,1)$ and $x, y \in H$.

In this paper, we consider the following Noor type iterative scheme and use it to obtain a strong convergence for an asymptotically pseudocontractive mappings in the intermediate sense.

Let $T : C \to C$ be a mapping. For an arbitrary $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty} \subset C$ defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T^n z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n \end{aligned} \tag{1.26}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$, are three sequences satisfying $\alpha_n, \beta_n, \gamma_n \in [0,1]$ for each n.

2 Strong convergence theorem for asymptotically pseudocontractive mappings in the intermediate sense

Theorem 2.1. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in the intermediate sense with sequences $\{k_n\} \subset [1,\infty)$ and $\{\tau_n\} \subset [0,\infty)$ defined as in (1.14). Assume that the interior of $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Assume that the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \tau_n < \infty$, $\sum_{n=1}^{\infty}(q_n^3 - 1) < \infty$ where $q_n := 2k_n - 1$ for each $n \geq 1$;
(ii) $a \leq \alpha_n \leq \beta_n \leq \gamma_n \leq b$ for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1+L^2} - 1])$.
Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to a fixed point of T.

Proof. Fix $p \in F(T)$. From Lemma 1.3, (2.1) and (1.17), we obtain

$$\begin{aligned} \|z_n - p\|^2 &= \|(1-\gamma_n)(x_n - p) + \gamma_n(T^n x_n - p)\|^2 \\ &= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|T^n x_n - p\|^2 - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2 \\ &\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\{q_n\|x_n - p\|^2 + \|x_n - T^n x_n\|^2 + 2\tau_n\} \\ &\quad -\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2 \\ &\leq q_n\|x_n - p\|^2 + \gamma_n\|x_n - T^n x_n\|^2 + 2\gamma_n\tau_n - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2 \\ &\leq q_n\|x_n - p\|^2 + \gamma_n^2\|T^n x_n - x_n\|^2 + 2\tau_n. \end{aligned} \tag{2.2}$$

$$\begin{aligned} \|z_n - T^n z_n\|^2 &= \|(1-\gamma_n)(x_n - T^n z_n) + \gamma_n(T^n x_n - T^n z_n)\|^2 \\ &= (1-\gamma_n)\|x_n - T^n z_n\|^2 + \gamma_n\|T^n x_n - T^n z_n\|^2 \\ &\quad -\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2 \\ &\leq (1-\gamma_n)\|x_n - T^n z_n\|^2 + \gamma_n^3 L^2\|x_n - T^n x_n\|^2 \\ &\quad -\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2. \end{aligned} \tag{2.3}$$

Using Lemma 1.3, (1.17), (2.1), (2.2) and (2.3), we obtain:

$$
\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n z_n - p)\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n z_n - p\|^2 - \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 \\
&\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\{q_n\|z_n - p\|^2 + \|z_n - T^n z_n\|^2 + 2\tau_n\} \\
&\quad -\beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 \\
&\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\{q_n(q_n\|x_n - p\|^2 + \gamma_n^2\|T^n x_n - x_n\|^2 + 2\tau_n) + \\
&\quad (1-\gamma_n)\|x_n - T^n z_n\|^2 + \gamma_n^3 L^2\|x_n - T^n x_n\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2 + 2\tau_n\} \\
&\le q_n^2\|x_n - p\|^2 + \beta_n q_n \gamma_n^2\|T^n x_n - x_n\|^2 + 2q_n\tau_n \\
&\quad +\beta_n(1-\gamma_n)\|x_n - T^n z_n\|^2 + \beta_n\gamma_n^3 L^2\|x_n - T^n x_n\|^2 - \\
&\quad \beta_n\gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2 + 2\tau_n \\
&\le q_n^2\|x_n - p\|^2 - \beta_n\gamma_n(1-\gamma_n-\gamma_n q_n - \gamma_n^2 L^2)\|T^n x_n - x_n\|^2 \\
&\quad +\beta_n(1-\gamma_n)\|x_n - T^n z_n\|^2 + 2\tau_n(1+q_n). \qquad (2.4)
\end{aligned}
$$

Using Lemma 1.3, (1.17), (2.1) and (2.3), we have

$$
\begin{aligned}
\|y_n - T^n y_n\|^2 &= \|(1-\beta_n)(x_n - T^n y_n) + \beta_n(T^n z_n - T^n y_n)\|^2 \\
&= (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n\|T^n z_n - T^n y_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 \\
&\le (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2\|z_n - T^n z_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 \\
&\le (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2\{(1-\gamma_n)\|x_n - T^n z_n\|^2 + \\
&\quad \gamma_n^3 L^2\|x_n - T^n x_n\|^2 - \gamma_n(1-\gamma_n)\|T^n x_n - x_n\|^2\} - \\
&\quad \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 \\
&= (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2(1-\gamma_n)\|x_n - T^n z_n\|^2 - \\
&\quad \beta_n^3 L^2\gamma_n(1-\gamma_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 - \\
&\quad \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2. \qquad (2.5)
\end{aligned}
$$

Using (1.17), (2.4) and (2.5), we have

$$
\begin{aligned}
\|T^n y_n - p\|^2 &\le q_n\|y_n - p\|^2 + \|y_n - T^n y_n\|^2 + 2\tau_n \\
&\le q_n\{q_n^2\|x_n - p\|^2 - \beta_n\gamma_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad \beta_n(1-\gamma_n)\|x_n - T^n z_n\|^2 + 2\tau_n(1+q_n)\} + \\
&\quad (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2(1-\gamma_n)\|x_n - T^n z_n\|^2 - \\
&\quad \beta_n^3 L^2\gamma_n(1-\gamma_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 - \\
&\quad \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 + 2\tau_n \\
&= q_n^3\|x_n - p\|^2 - \beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad \beta_n q_n(1-\gamma_n)\|x_n - T^n z_n\|^2 + 2q_n\tau_n(1+q_n) + \\
&\quad (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n^3 L^2(1-\gamma_n)\|x_n - T^n z_n\|^2 - \\
&\quad \beta_n^3 L^2\gamma_n(1-\gamma_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 - \\
&\quad \beta_n(1-\beta_n)\|T^n z_n - x_n\|^2 + 2\tau_n \\
&\le q_n^3\|x_n - p\|^2 - \beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad (1-\beta_n)\|x_n - T^n y_n\|^2 + 2\tau_n(1+q_n+q_n^2). \qquad (2.6)
\end{aligned}
$$

Using Lemma 1.3, (1.17) and (2.6), we obtain:

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n y_n - p)\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\{q_n^3\|x_n - p\|^2 - \\
&\quad \beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad (1-\beta_n)\|x_n - T^n y_n\|^2 + 2\tau_n(1+q_n+q_n^2)\} - \\
&\quad \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\
&\le q_n^3\|x_n - p\|^2 - \alpha_n\beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad \alpha_n(1-\beta_n)\|x_n - T^n y_n\|^2 + 2\tau_n(1+q_n+q_n^2) - \\
&\quad \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\
&\le q_n^3\|x_n - p\|^2 - \alpha_n\beta_n\gamma_n q_n(1-\gamma_n-\gamma_n q_n-\gamma_n^2 L^2)\|T^n x_n - x_n\|^2 + \\
&\quad 2\tau_n(1+q_n+q_n^2). \qquad (2.7)
\end{aligned}
$$

From assumption (ii) $0 < \alpha_n \le \beta_n$ implies that $0 < \alpha_n(1-\beta_n) < \alpha_n(1-\alpha_n)$ and $\|x_n - T^n y_n\|^2 =$

$\|(-1)(x_n - T^n y_n)\|^2 = |-1|^2\|x_n - T^n y_n\|^2 = \|x_n - T^n y_n\|^2$. So that $\alpha_n(1-\beta_n)\|x_n - T^n y_n\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n y_n\|^2 = -k\|x_n - T^n y_n\|^2$ for some constant $k > 0$. Hence, we obtain (2.7).
Observe from condition (ii) $b \in (0, L^{-2}[\sqrt{1+L^2}-1])$ implies that $b > 0$ and $b < L^{-2}[\sqrt{1+L^2}-1]$. This implies that $bL^2 < \sqrt{1+L^2}-1$, hence $1 + bL^2 < \sqrt{1+L^2}$. On squaring both sides, we obtain $(1+bL^2)^2 < (\sqrt{1+L^2})^2$, so that $1 + 2bL^2 + b^2L^4 < 1 + L^2$, so we obtain $L^2 - 2bL^2 - b^2L^4 > 0$, by dividing through by L^2, we obtain $1 - 2b - b^2L^2 > 0$. Hence, $\frac{1-2b-b^2L^2}{3} > 0$.
Inview of the fact that $\gamma_n \leq b$ and condition (ii), there exists n_0 such that

$$1 - \gamma_n - \gamma_n q_n - \gamma_n^2 L^2 \geq \frac{1-2b-L^2b^2}{3} > 0, \quad \forall n \geq n_0, \tag{2.8}$$

hence, (2.7) gives

$$\|x_{n+1} - p\|^2 \leq \{1 + (q_n^3 - 1)\}\|x_n - p\|^2 + 2\tau_n(1 + q_n + q_n^2), \ \ \forall n \geq n_0. \tag{2.9}$$

Hence, by Lemma 1.2 we have that $\lim_{n\to\infty} \|x_n - p\|$ exists.

Using (1.23), we obtain

$$\phi(p, x_n) = \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, x_n - x_{n+1}\rangle. \tag{2.10}$$

This implies

$$\langle x_{n+1} - p, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}\left\{\phi(p, x_n) - \phi(p, x_{n+1})\right\}. \tag{2.11}$$

Since the interior of $F(T)$ is nonempty, there exists $x^* \in F(T)$ and $r > 0$ such that $x^* + rh \in F(T)$, whenever $\|h\| \leq 1$. Hence, by replacing p with $x^* + rh$ in (2.10) and using it in (2.11) and by using assumption (i), we have

$$0 \leq \langle x_{n+1} - (x^* + rh), x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + M((q_n^3 - 1) + \tau_n), \tag{2.12}$$

for some $M > 0$. Consequently, from (2.11) and (2.12) we have that

$$\begin{aligned} r\langle h, x_n - x_{n+1}\rangle &\leq \langle x_{n+1} - x^*, x_n - x_{n+1}\rangle + \tfrac{1}{2}\phi(x_{n+1}, x_n) + M((q_n^3 - 1) + \tau_n) \\ &= \tfrac{1}{2}(\phi(x^*, x_n) - \phi(x^*, x_{n+1})) + M((q_n^3 - 1) + \tau_n), \end{aligned} \tag{2.13}$$

hence,

$$\langle h, x_n - x_{n+1}\rangle \leq \frac{1}{2r}(\phi(x^*, x_n) - \phi(x^*, x_{n+1})) + \frac{1}{r}M((q_n^3 - 1) + \tau_n). \tag{2.14}$$

But h with $\|h\| \leq 1$ is arbitrary, we obtain

$$\|x_n - x_{n+1}\| \leq \frac{1}{2r}(\phi(x^*, x_n) - \phi(x^*, x_{n+1})) + \frac{1}{r}M((q_n^3 - 1) + \tau_n). \tag{2.15}$$

Hence, if $n > m > n_0$, we obtain

$$\begin{aligned} \|x_m - x_n\| &= \|x_m - x_{m+1} + x_{m+1} - \cdots - x_{n-1} + x_{n-1} - x_n\| \\ &\leq \textstyle\sum_{i=m}^{n-1} \|x_i - x_{i+1}\| \\ &\leq \tfrac{1}{2r}\textstyle\sum_{i=m}^{n-1}(\phi(x^*, x_i) - \phi(x^*, x_{i+1})) + \tfrac{M}{r}\textstyle\sum_{i=m}^{n-1}((q_i^3 - 1) + \tau_i) \\ &= \tfrac{1}{2r}(\phi(x^*, x_m) - \phi(x^*, x_n)) + \tfrac{M}{r}\textstyle\sum_{i=m}^{n-1}((q_i^3 - 1) + \tau_i). \end{aligned} \tag{2.16}$$

But $\{\phi(x^*, x_m)\}$ converges, $\sum \tau_n < \infty$ and $\sum(q_n^3 - 1) < \infty$. Hence, we have that $\{x_n\}$ is a Cauchy sequence. But H is complete, this implies that there exists $y^* \in H$ such that

$$x_n \to y^* \in H. \tag{2.17}$$

Since $\{x_n\}$ is a subset of C which is closed and convex we have that $y^* \in C$. Since C is complete, we claim that $y^* \in F(T)$. Using (2.7) and (2.8), we obtain

$$\frac{a^2(1-2b-L^2b^2)}{3}\|T^n x_n - x_n\|^2 \leq q_n^3\|x_n - y^*\|^2 - \|x_{n+1} - y^*\|^2 + 2\tau_n(1 + q_n + q_n^2). \tag{2.18}$$

Hence,

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{2.19}$$

Since $x_n \to y^*$ we obtain $T^n x_n \to y^*$ as $n \to \infty$.

Next we, show that $\|T^n y^* - y^*\| \to 0$ as $n \to \infty$. Recall that T is uniformly L-Lipschitzian and $x_n \to y^*$ as $n \to \infty$, we obtain

$$\|T^n y^* - T^n x_n\| \leq L\|y^* - x_n\| \to 0, \quad as\ n \to \infty, \tag{2.20}$$

hence,

$$T^n y^* \to y^* \quad as\ n \to \infty. \tag{2.21}$$

Consequently, by continuity of T we obtain $y^* = \lim_{n\to\infty}(T^n y^*) = \lim_{n\to\infty} T(T^{n-1}y^*) = T(\lim_{n\to\infty}(T^{n-1}y^*)) = T(y^*)$, meaning that $y^* \in F(T)$. The proof of the theorem is complete.

We obtain the following corollaries to Theorem 2.1.

Corollary 2.2. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be uniformly L-Lipschitzian and *asymptotically pseudocontractive* mappings with sequences $\{k_n\} \subset [1, \infty)$. Assume that the interior of $F(T)$ is nonempty. Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to a fixed point of T.

Proof. Let $\tau_n = 0$ for all $n \geq 1$ in Theorem 2.1, and the proof follows.
If we assume that T is *asymptotically nonexpansive* in corollary 2.2, then we obtain the following corollary.

Corollary 2.3. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be *asymptotically nonexpansive* mappings with sequences $\{k_n\} \subset [1, \infty)$. Assume that the interior of $F(T)$ is nonempty. Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to a common fixed point of T.

Proof. Recall that every asymptotically nonexpansive mappings is uniformly L-Lipschitzian with $L := \max_{n\geq 1}\{k_n\}$ and asymptotically pseudocontractive mapping, hence the proof follows from corollary 2.2.

Remark 2.4. If $\gamma_n = 0\ \ \forall n \geq 1$ in Theorem 2.3 we obtain Theorem ZRC which is an improvement of Theorem QCK since the Noor type iterative scheme we used is more general than the Ishikawa type iterative scheme used in Theorem QCK and Schu (1991). Our convergence is strong and does not require the complex computation of $C_n \cap Q_n$ for each $n \geq 1$ as was the case of Qin *et al.* (2010). Corollary 2.3 extends the results of Schu (1991) in the sense that our resuts does not require that T be completely continuous or C be compact.

3 Strong convergence theorem for asymptotically strict pseudocontractive mappings in the intermediate sense

Theorem 3.1. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be a uniformly L-Lipschitzian and *asymptotically strict pseudocontractive mapping in the intermediate sense* with sequences $\{k_n\} \subset [1, \infty)$ and $\{\zeta_n\} \subset [0, \infty)$ as defined in (1.9). Assume that the interior of $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n z_n \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$. Assume that the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \zeta_n < \infty, \ \sum_{n=1}^{\infty} (k_n^3 - 1) < \infty$
(ii) $a \le \alpha_n \le \beta_n \le \gamma_n \le b$ for some $a > 0$ and some $b \in (0, L^{-2}[\sqrt{1+L^2} - 1])$.
Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a fixed point of T.

Proof. Observe that any L-Lipschitzian and asymptotically k-strict pseudocontractive mapping T in the intermediate sense is uniformly L-Lipschitzian and asymptotically pseudocontractive mapping in the intermediate sense with $q_n := k_n$ and $\tau_n := \frac{1}{2}\zeta_n \ \forall n \ge 1$, consequently, the conclusion follows from Theorem 2.1.

Corollary 3.2. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be an asymptotically strict pseudocontractive mapping with sequences $\{k_n\} \subset [1, \infty)$. Assume that the interior of $F(T)$ is nonempty. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a fixed point of T.

Proof. Recall that any k-strict pseudocontractive mapping T is uniformly L-Lipschitzian, since $\|T^n x - T^n y\| \le L\|x - y\|, \ \forall x, y \in C$, where $L = \max\{\frac{k+\sqrt{1+(k_n-1)(1-k)}}{1-k}\}$ (Kim and Xu (2008)). Hence, the proof follows from Theorem 3.1 with $\zeta_n = 0$ for all $n \ge 1$.

Remark 3.3. Observe that Corollary 3.2 extends Theorems 3.1 and 4.1 of Kim and Xu (2008), Qin *et al.* (2010) and Corollary 3.2 of Zegeye *et al.* (2011) in the sense that we obtained a strong convergence and do not require the computation of $C_n \cap Q_n$ for all $n \ge 1$. If we take $\gamma_n = 0 \ \forall n \ge 1$, then we obtain Corollary 3.2 of Zegeye *et al.* (2011).

Example 3.4. Let $X = \mathbb{R}$ and $C = [0, 1]$, for each $x \in C$. Define

$$Tx = \begin{cases} e^{-\sqrt{k}}x, & if \ \ x \in [0, \frac{1}{2}] \\ \\ 0, & if \ \ x \in (\frac{1}{2}, 1] \end{cases} \tag{3.2}$$

where $0 < k < 1$. Then $T : C \to C$ is not continuous at $x = \frac{1}{2}$, this impies that T is not Lipschitzian. Set $C_1 := [1, \frac{1}{2}]$ and $C_2 := (\frac{1}{2}, 1]$. Hence, we obtain
$|T^n x - T^n y| = e^{-\sqrt{k}n}|x - y| \le |x - y|$ for all $x, y \in C_1$ and $n \in \mathbb{N}$. and
$|T^n x - T^n y| = 0 \le |x - y|$ for each $x, y \in C_2$ and $n \in \mathbb{N}$.
For $x \in C_1$ and $y \in C_2$, we obtain:

$$\begin{aligned} |T^n x - T^n y| &= |e^{-\sqrt{k}n}x - 0| = |e^{-\sqrt{k}n}(x - y) + e^{-\sqrt{k}n}y| \\ &\le e^{-\sqrt{k}n}|x - y| + e^{-\sqrt{k}n}|y| \\ &\le |x - y| + e^{-\sqrt{k}n} \ \ \forall n \in \mathbb{N}. \end{aligned} \tag{3.3}$$

Thus,

$$\begin{aligned} |T^n x - T^n y|^2 &\le (|x - y| + e^{-\sqrt{k}n})^2 \\ &\le |x - y| + e^{-\sqrt{k}}|x - T^n x - (y - T^n y)|^2 + e^{-\sqrt{k}n}M, \end{aligned} \tag{3.4}$$

for each $x, y \in C, \ n \in \mathbb{N}$ and for some $M > 0$.
Hence, T is an asymptotically k-strict pseudocontractive mapping in the intermediate sense.

Remark 3.5. Observe that since T is not continuous, T is not asymptotically k-strictly pseudocontractive and asymptotically nonexpansive in the intermediate sense.

Authors' contributions

All authors contributed equally and significantly in this research work. All authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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